

MICRO-LOCAL ANALYSIS WITH FOURIER LEBESGUE SPACES. PART I

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ABSTRACT. Let ω, ω_0 be appropriate weight functions and $q \in [1, \infty]$. We introduce the wave-front set, $\text{WF}_{\mathcal{FL}^q_{(\omega)}}(f)$ of $f \in \mathcal{S}'$ with respect to weighted Fourier Lebesgue space $\mathcal{FL}^q_{(\omega)}$. We prove that usual mapping properties for pseudo-differential operators $\text{Op}(a)$ with symbols a in $S^{(\omega_0)}_{\rho,0}$ hold for such wave-front sets. Especially we prove that

$$\begin{aligned} \text{WF}_{\mathcal{FL}^q_{(\omega/\omega_0)}}(\text{Op}(a)f) &\subseteq \text{WF}_{\mathcal{FL}^q_{(\omega)}}(f) \\ &\subseteq \text{WF}_{\mathcal{FL}^q_{(\omega/\omega_0)}}(\text{Op}(a)f) \cup \text{Char}(a). \quad (*) \end{aligned}$$

Here $\text{Char}(a)$ is the set of characteristic points of a .

0. INTRODUCTION

In this paper we introduce wave-front sets of appropriate Banach (and Fréchet) spaces. We especially consider the case when these Banach spaces are Fourier-Lebesgue type spaces. The family of such wave-front sets contains the wave-front sets of Sobolev type, introduced by Hörmander in [17], as well as the classical wave-front sets with respect to smoothness (cf. Sections 8.1 and 8.2 in [16]), as special cases. Roughly speaking, for any given distribution f and for appropriate Banach (or Fréchet) space \mathcal{B} of tempered distributions, the wave-front set $\text{WF}_{\mathcal{B}}(f)$ of f consists of all pairs (x_0, ξ_0) in $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that no localizations of the distribution at x_0 belongs to \mathcal{B} in the direction ξ_0 .

We also establish mapping properties for a quite general class of pseudo-differential operators on such wave-front sets, and show that our approach leads to a flexible micro-local analysis tools which fits well to the most common approach developed in e.g. [16, 17]. Especially we prove that usual mapping properties, which are valid for classical wave-front sets (cf. Chapters VIII and XVIII in [16]) also hold for wave-front sets of Fourier-Lebesgue type. For example, we prove (*) in the abstract, that is, any operator $\text{Op}(a)$ to some extent shrink the wave-front sets and the opposite embedding can be obtained by including $\text{Char}(a)$, the set of characteristic points of the operator symbol a .

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The symbol classes for the pseudo-differential operators are of the form $S_{\rho,\delta}^{(\omega_0)}(\mathbf{R}^{2d})$ which consists of all smooth functions a on \mathbf{R}^{2d} such that $a/\omega_0 \in S_{\rho,\delta}^0(\mathbf{R}^{2d})$. Here $\rho, \delta \in \mathbf{R}$ and ω_0 is an appropriate smooth function on \mathbf{R}^{2d} . We note that $S_{\rho,\delta}^{(\omega_0)}(\mathbf{R}^{2d})$ agrees with the Hörmander class $S_{\rho,\delta}^r(\mathbf{R}^{2d})$ when $\omega_0(x, \xi) = \langle \xi \rangle^r$, where $r \in \mathbf{R}$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

The set of characteristic points $\text{Char}(a)$ of $a \in S_{\rho,\delta}^{(\omega_0)}$ depends on the choices of ρ , δ and ω_0 (see Definition 1.3). This set is empty when a satisfies a local ellipticity condition with respect to ω_0 . In contrast to Section 18.1 in [16], $\text{Char}(a)$ is defined for all symbols in $S_{\rho,\delta}^{(\omega_0)}$, and not only for polyhomogeneous symbols. Furthermore, if a is a polyhomogeneous symbol, then $\text{Char}(a)$ is smaller than the set of characteristic points, given by [16] (see Remark 1.4 and Example 3.9). This is especially demonstrated for a broad class of hypoelliptic partial differential operators. For any hypoelliptic operator $\text{Op}(a)$ with constant coefficients and with symbol a , we may choose the symbol class such that it contains a , and such that a is elliptic with respect to that weight. Consequently, $\text{Char}(a)$ is empty, and in view of (*) in the abstract it follows that such hypoelliptic operators preserve the wave-front sets, as they should (see Theorems 3.7 and 4.5, and Corollary 3.8).

Information on regularity in the background of wave-front sets of Fourier Lebesgue types might be more detailed comparing to classical wave-front sets, because we may play with the exponent $q \in [1, \infty]$ and the weight function ω in our choice of Fourier Lebesgue space $\mathcal{FL}_{(\omega)}^q(\mathbf{R}^d)$. By choosing $q = 1$ and $\omega(\xi) = \langle \xi \rangle^N$, where $N \geq 0$ is an integer, $\mathcal{FL}_{(\omega)}^1(\mathbf{R}^d)$ locally contains $C^{N+d+1}(\mathbf{R}^d)$, and is contained in $C^N(\mathbf{R}^d)$. Consequently, our wave-front sets can be used to investigate micro-local properties which, in some sense, are close to C^N -regularity.

Another example is obtained by choosing $q = \infty$ and $\omega = \omega_0$. If E is a parametrix to a pseudo-differential operator $\text{Op}(a)$ with $a \in S_{\rho,0}^{(\omega_0)}$, then

$$\text{Op}(a)E = \delta_0 + \varphi,$$

which belongs locally to \mathcal{FL}^∞ , giving that $\text{WF}_{\mathcal{FL}^\infty}(\text{Op}(a)E)$ is empty. Hence (*) in the abstract shows that $\text{WF}_{\mathcal{FL}_{(\omega)}^\infty}(E)$ is contained in $\text{Char}(a)$. In particular, if in addition $\text{Op}(a)$ is elliptic with respect to ω_0 , then it follows that $\text{WF}_{\mathcal{FL}_{(\omega)}^\infty}(E)$ is empty, or equivalently, E is locally in $\mathcal{FL}_{(\omega)}^\infty$. This implies that for each $x \in \mathbf{R}^d$ and test function φ on \mathbf{R}^d we have

$$|\mathcal{F}(\varphi E)(\xi)| \leq C\omega(x, \xi)^{-1}, \quad (0.1)$$

for some constant C . Here we remark that every hypoelliptic partial differential operator with constant coefficients is elliptic with respect to some admissible weight ω_0 . Therefore, our results can be applied in

efficient ways on such operators. (See Theorem 3.7 and Corollary 3.8 for the details.)

In the second part of the paper (Sections 5 and 6) we define wave-front sets with respect to (weighted) modulation spaces (which also involve certain types of Wiener amalgam spaces), and prove that they coincide with the wave-front sets of Fourier Lebesgue type. Here we also extend some wave-front results to pseudo-differential operators with symbols which are defined in terms of modulation spaces of "weighted Sjöstrand type". These symbol classes are superclasses to $S_{\rho,0}^{(\omega_0)}(\mathbf{R}^{2d})$, and contain non-smooth symbols.

The modulation spaces have been introduced by Feichtinger in [5], and the theory was developed further and generalized in [7–9, 11]. The modulation space $M_{(\omega)}^{p,q}(\mathbf{R}^d)$, where ω denotes a weight function on phase (or time-frequency) space \mathbf{R}^{2d} , is the set of tempered (ultra-) distributions whose short-time Fourier transform belongs to the weighted and mixed Lebesgue space $L_{(\omega)}^{p,q}(\mathbf{R}^{2d})$. It follows that the weight ω quantifies the degrees of asymptotic decay and singularity of the distributions.

Modulation spaces have been, in parallel, incorporated into the calculus of pseudo-differential operators, through the study of continuity of (classical) pseudo-differential operators acting on modulation spaces (cf. [4, 19, 20, 24–26]), as well as through the analysis of operators of non-classical type, where modulation spaces are used as symbol classes. For example, after a systematic development of the modulation space theory already had been done by Feichtinger and Gröchenig, Sjöstrand introduced in [23] a superspace of $S_{0,0}^0$ which turned out to coincide with $M^{\infty,1}$, and used this modulation space as a symbol class. He proved that $M^{\infty,1}$ as symbol class corresponds to an algebra of operators which are bounded on L^2 . Sjöstrand's results were thereafter further extended in [12–14, 27–29].

The paper is organized as follows. In the beginning of Section 1 we recall the definition of (weighted) Fourier Lebesgue spaces. We continue with the definition and basic properties of pseudo-differential operators in Subsection 1.1. Then, in Subsection 1.2 we define sets of characteristic points for a broad class of pseudo-differential operators and prove that these sets might be smaller than characteristic sets in [16] (see also Example 3.9 in Section 3). In Subsection 1.3 we recall the definition and basic properties of modulation spaces, and, in Subsection 1.4 we introduce a class of pseudo-differential operators with non-smooth symbols in the context of modulation spaces. In Section 2 we define wave-front sets with respect to (weighted) Fourier Lebesgue spaces, and prove some important properties for such wave-front sets. Thereafter we consider in Section 3 mapping properties for pseudo-differential operators

in context of these wave-front sets, and, in particular, we prove (*) in the abstract.

In Section 4 we consider wave-front sets obtained from sequences of Fourier Lebesgue space spaces. We show that these types of wave-front sets contain classical wave-front sets (with respect to smoothness), and that the mapping properties for pseudo-differential operators also hold in context of such wave-front sets. In particular, we recover the well-known property (*) in the abstract, for usual wave-front sets (cf. Section 18.1 in [16]).

In Section 5 we introduce wave-front sets with respect to modulation spaces, and prove that they coincide with wave-front sets of Fourier Lebesgue types. In Section 6 we consider mapping properties on wave-front sets of pseudo-differential operators with symbol classes defined in terms of weighted Sjöstrand classes.

Finally, we remark that the present paper is the first one in series of papers. In the second paper [21] we consider products in Fourier Lebesgue spaces, related to the new notion of wave-fronts with applications to a class of semilinear partial differential equations. In [18] the authors together with Karoline Johansson show that the wave-front sets of Fourier Lebesgue and modulation space types can be discretized, and how they can be implemented in numerical computations.

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1. PRELIMINARIES

In this section we recall some notations and basic results. In what follows we let Γ denote an open cone in $\mathbf{R}^d \setminus 0$ with vertex at origin. If $\xi \in \mathbf{R}^d \setminus 0$ is fixed, then an open cone which contains ξ is sometimes denoted by Γ_ξ .

Assume that ω and v are positive and measurable functions on \mathbf{R}^d . Then ω is called v -moderate if

$$\omega(x+y) \leq C\omega(x)v(y) \quad (1.1)$$

for some constant C which is independent of $x, y \in \mathbf{R}^d$. If v in (1.1) can be chosen as a polynomial, then ω is called polynomially moderated. We let $\mathcal{P}(\mathbf{R}^d)$ be the set of all polynomially moderated functions on \mathbf{R}^d . If $\omega(x, \xi) \in \mathcal{P}(\mathbf{R}^{2d})$ is constant with respect to the x -variable (ξ -variable), then we sometimes write $\omega(\xi)$ ($\omega(x)$) instead of $\omega(x, \xi)$.

In this case we consider ω as an element in $\mathcal{P}(\mathbf{R}^{2d})$ or in $\mathcal{P}(\mathbf{R}^d)$ depending on the situation.

We also need to consider classes of weight functions, related to \mathcal{P} . More precisely, we let $\mathcal{P}_0(\mathbf{R}^d)$ be the set of all $\omega \in \mathcal{P}(\mathbf{R}^d) \cap C^\infty(\mathbf{R}^d)$ such that $\partial^\alpha \omega / \omega \in L^\infty$ for all multi-indices α . For each $\omega \in \mathcal{P}(\mathbf{R}^d)$, there is an equivalent weight $\omega_0 \in \mathcal{P}_0(\mathbf{R}^d)$, that is, $C^{-1}\omega_0 \leq \omega \leq C\omega_0$ holds for some constant C (cf. [28, Lemma 1.2]).

Assume that $\rho, \delta \in \mathbf{R}$. Then we let $\mathcal{P}_{\rho, \delta}(\mathbf{R}^{2d})$ be the set of all $\omega(x, \xi)$ in $\mathcal{P}(\mathbf{R}^{2d}) \cap C^\infty(\mathbf{R}^{2d})$ such that

$$\langle \xi \rangle^{\rho|\beta| - \delta|\alpha|} \frac{\partial_x^\alpha \partial_\xi^\beta \omega(x, \xi)}{\omega(x, \xi)} \in L^\infty(\mathbf{R}^{2d}),$$

for every multi-indices α and β . Note that in contrast to \mathcal{P}_0 , we do not have an equivalence between $\mathcal{P}_{\rho, \delta}$ and \mathcal{P} when $\rho > 0$. On the other hand, if $s \in \mathbf{R}$ and $\rho \in [0, 1]$, then $\mathcal{P}_{\rho, \delta}(\mathbf{R}^{2d})$ contains $\omega(x, \xi) = \langle \xi \rangle^s$, which are one of the most important classes in the applications.

The Fourier transform \mathcal{F} is the linear and continuous mapping on $\mathcal{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) e^{-i\langle x, \xi \rangle} dx$$

when $f \in L^1(\mathbf{R}^d)$. The map \mathcal{F} is a homeomorphism on $\mathcal{S}'(\mathbf{R}^d)$ which restricts to a homeomorphism on $\mathcal{S}(\mathbf{R}^d)$ and to a unitary operator on $L^2(\mathbf{R}^d)$.

Let $q \in [1, \infty]$ and $\omega \in \mathcal{P}(\mathbf{R}^d)$. The (weighted) Fourier Lebesgue space $\mathcal{F}L_{(\omega)}^q(\mathbf{R}^d)$ is the inverse Fourier image of $L_{(\omega)}^q(\mathbf{R}^d)$, i. e. $\mathcal{F}L_{(\omega)}^q(\mathbf{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{F}L_{(\omega)}^q} \equiv \|\widehat{f} \cdot \omega\|_{L^q}. \quad (1.2)$$

is finite. If $\omega = 1$, then the notation $\mathcal{F}L^q$ is used instead of $\mathcal{F}L_{(\omega)}^q$. We note that if $\omega(\xi) = \langle \xi \rangle^s$, then $\mathcal{F}L_{(\omega)}^q$ is the Fourier image of the Bessel potential space H_s^p (cf. [1]).

Here and in what follows we use the notation $\mathcal{F}L_{(\omega)}^q$ instead of the less cumbersome $\mathcal{F}L_\omega^q$, because in forthcoming papers (cf. [18, 21]), we often assume that ω is of the particular form $\omega(\xi) = \langle \xi \rangle^s$, and in this case we set $\mathcal{F}L_s^q = \mathcal{F}L_{(\omega)}^q$ without brackets for the weight parameter.

Remark 1.1. In many situations it is convenient to permit an x dependency for the weight ω in the definition of Fourier Lebesgue spaces. More precisely, for each $\omega \in \mathcal{P}(\mathbf{R}^{2d})$ we let $\mathcal{F}L_{(\omega)}^q$ be the set of all $f \in \mathcal{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{F}L_{(\omega)}^q} \equiv \|\widehat{f} \omega(x, \cdot)\|_{L^q}$$

is finite. Since ω is v -moderate for some $v \in \mathcal{P}(\mathbf{R}^{2d})$ it follows that different choices of x give rise to equivalent norms. Therefore the condition $\|f\|_{\mathcal{F}L_{(\omega)}^q} < \infty$ is independent of x , and it follows that $\mathcal{F}L_{(\omega)}^q(\mathbf{R}^d)$ is independent of x although $\|\cdot\|_{\mathcal{F}L_{(\omega)}^q}$ might depend on x .

1.1. Pseudodifferential operators. In this subsection we recall some facts from Chapter XVIII in [16] concerning pseudo-differential operators. Let $a \in \mathcal{S}(\mathbf{R}^{2d})$, and let $t \in \mathbf{R}$ be fixed. Then the pseudo-differential operator $\text{Op}_t(a)$ which corresponds to the symbol a is the linear and continuous operator on $\mathcal{S}(\mathbf{R}^d)$, defined by

$$(\text{Op}_t(a)f)(x) = (2\pi)^{-d} \iint a((1-t)x + ty, \xi) f(y) e^{i\langle x-y, \xi \rangle} dy d\xi. \quad (1.3)$$

If $t = 0$, then $\text{Op}_t(a)$ is the Kohn-Nirenberg representation $a(x, D) = \text{Op}(a) = \text{Op}_0(a)$, and if $t = 1/2$, then $\text{Op}_t(a)$ is the Weyl quantization of a .

For general $a \in \mathcal{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $\text{Op}_t(a)$ is defined as the operator with distribution kernel

$$K_{t,a}(x, y) = (2\pi)^{-d/2} (\mathcal{F}_2^{-1}a)((1-t)x + ty, x - y). \quad (1.4)$$

Here $\mathcal{F}_2 F$ is the partial Fourier transform of $F(x, y) \in \mathcal{S}'(\mathbf{R}^{2d})$ with respect to the y -variable. We remark that $K_{t,a}$ makes sense as a distribution in $\mathcal{S}'(\mathbf{R}^{2d})$. In fact, the map

$$F(x, y) \mapsto F((1-t)x + ty, x - y)$$

is obviously a homeomorphism on $\mathcal{S}(\mathbf{R}^{2d})$ and on $\mathcal{S}'(\mathbf{R}^{2d})$. Furthermore, by straight-forward computations it follows that the partial Fourier transform \mathcal{F}_2 is a homeomorphism on $\mathcal{S}(\mathbf{R}^{2d})$, and extends in the usual way to a homeomorphism on $\mathcal{S}'(\mathbf{R}^{2d})$ which is unitary on L^2 (cf. e.g. Section 7.1 in [16]). Consequently, if $a \in \mathcal{S}'(\mathbf{R}^{2d})$, then $K_{t,a}$ in (1.4) makes sense as a tempered distribution, and $\text{Op}_t(a)$ is a continuous operator from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$. As a consequence of Schwartz kernel theorem it follows that the map $a \mapsto \text{Op}_t(a)$ is bijective from $\mathcal{S}'(\mathbf{R}^{2d})$ to the set of linear and continuous operators from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$.

We also note that $K_{t,a}$ belongs to $\mathcal{S}(\mathbf{R}^{2d})$, if and only if $a \in \mathcal{S}(\mathbf{R}^{2d})$, and that the latter definition of $\text{Op}_t(a)$ agrees with the operator in (1.3) when $a \in \mathcal{S}(\mathbf{R}^{2d})$.

If $a \in \mathcal{S}'(\mathbf{R}^{2d})$ and $s, t \in \mathbf{R}$, then there is a unique $b \in \mathcal{S}'(\mathbf{R}^{2d})$ such that $\text{Op}_s(a) = \text{Op}_t(b)$. By straight-forward applications of Fourier's inversion formula, it follows that

$$\text{Op}_s(a) = \text{Op}_t(b) \iff b(x, \xi) = e^{i(t-s)\langle D_x, D_\xi \rangle} a(x, \xi). \quad (1.5)$$

(Cf. Section 18.5 in [16].)

Next we discuss symbol classes which will be used in the sequel. Let $\rho, \delta \in \mathbf{R}$ and let $\omega_0 \in \mathcal{P}_{\rho, \delta}(\mathbf{R}^{2d})$. Then the symbol class $S_{\rho, \delta}^{(\omega_0)}(\mathbf{R}^{2d})$

consists of all $a \in C^\infty(\mathbf{R}^{2d})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} \omega_0(x, \xi) \langle \xi \rangle^{-\rho|\beta| + \delta|\alpha|}. \quad (1.6)$$

It is clear that $S_{\rho, \delta}^{(\omega_0)}$ is a Frechét space with semi-norms given by the smallest constant which can be used in (1.6).

If $\omega_0(x, \xi) = \langle \xi \rangle^r$, then $S_{\rho, \delta}^{(\omega_0)}(\mathbf{R}^{2d})$ agrees with the Hörmander class $S_{\rho, \delta}^r(\mathbf{R}^{2d})$. Usually it is assumed that $0 \leq \delta < \rho \leq 1$.

The following result shows that pseudo-differential operators with symbols in $S_{\rho, \delta}^{(\omega_0)}$ behave well.

Proposition 1.2. *Let $a \in S_{\rho, \delta}^{(\omega_0)}(\mathbf{R}^{2d})$ where $\omega_0 \in \mathcal{P}_{\rho, \delta}(\mathbf{R}^{2d})$, $\rho, \delta \in [0, 1]$, $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. Then $\text{Op}_t(a)$ is continuous on $\mathcal{S}(\mathbf{R}^d)$ and extends uniquely to a continuous operator on $\mathcal{S}'(\mathbf{R}^d)$.*

Proof. We have $S_{\rho, \delta}^{(\omega_0)}(\mathbf{R}^{2d}) = S(\omega_0, g_{\rho, \delta})$, when $g = g_{\rho, \delta}$ is the Riemannian metric on \mathbf{R}^{2d} , defined by the formula

$$(g_{\rho, \delta})_{(y, \eta)}(x, \xi) = \langle \eta \rangle^{2\delta} |x|^2 + \langle \eta \rangle^{-2\rho} |\xi|^2$$

(cf. Section 18.4–18.6 in [16]). From the assumptions it follows that $g_{\rho, \delta}$ is slowly varying, σ -temperate and satisfies $g_{\rho, \delta} \leq g_{\rho, \delta}^\sigma$, and that ω_0 is $g_{\rho, \delta}$ -continuous and σ , $g_{\rho, \delta}$ -temperate (cf. e.g. Sections 18.4–18.6 in [16] for definitions). The result is now a consequence of Proposition 18.5.10 and Theorem 18.6.2 in [16]. The proof is complete. \square

1.2. Sets of characteristic points. In this subsection we define the set of characteristic points of a symbol $a \in S_{\rho, \delta}^{(\omega_0)}(\mathbf{R}^{2d})$, when $\omega_0 \in \mathcal{P}_{\rho, \delta}(\mathbf{R}^{2d})$ and $0 \leq \delta < \rho \leq 1$. As remarked in the introduction, this definition is slightly different comparing to [16, Definition 18.1.5] in view of Remark 1.4 below.

Let $R > 0$, $X \subseteq \mathbf{R}^d$ be open and let $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone. For convenience we say that an element $c \in S_{\rho, \delta}^0(\mathbf{R}^{2d})$ is (X, Γ, R) -unitary when

$$c(x, \xi) = 1 \quad \text{when } x \in X, \xi \in \Gamma \text{ and } |\xi| \geq R.$$

If $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, then the element c is called *unitary near* (x_0, ξ_0) if c is (X, Γ, R) -unitary for some open neighbourhood X of x_0 , open conical neighbourhood Γ of ξ_0 and $R > 0$.

Definition 1.3. Let $0 \leq \delta < \rho \leq 1$, $\omega_0 \in \mathcal{P}_{\rho, \delta}(\mathbf{R}^{2d})$, $a \in S_{\rho, \delta}^{(\omega_0)}(\mathbf{R}^{2d})$, and set $\mu = \rho - \delta > 0$. The point (x_0, ξ_0) in $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ is called *non-characteristic* for a (with respect to ω_0), if there are elements $b \in S_{\rho, \delta}^{(1/\omega_0)}(\mathbf{R}^{2d})$, $c \in S_{\rho, \delta}^0(\mathbf{R}^{2d})$ which is unitary near (x_0, ξ_0) , and $h \in S_{\rho, \delta}^{-\mu}(\mathbf{R}^{2d})$ such that

$$b(x, \xi)a(x, \xi) = c(x, \xi) + h(x, \xi), \quad (x, \xi) \in \mathbf{R}^{2d}.$$

The point (x_0, ξ_0) in $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ is called *characteristic* for a (with respect to ω_0), if it is *not* non-characteristic for a with respect to ω_0 . The set of characteristic points (the characteristic set), for $a \in S_{\rho, \delta}^{(\omega_0)}(\mathbf{R}^{2d})$ with respect to ω_0 , is denoted by $\text{Char}(a) = \text{Char}_{(\omega_0)}(a)$.

Remark 1.4. Let $\omega_0(x, \xi) = \langle \xi \rangle^r$, $r \in \mathbf{R}$, and assume that $a \in S_{1,0}^r(\mathbf{R}^{2d}) = S_{1,0}^{(\omega_0)}(\mathbf{R}^{2d})$ is polyhomogeneous with principal symbol $a_r \in S_{1,0}^r(\mathbf{R}^{2d})$. (Cf. Definition 18.1.5 in [16].) Also let $\text{Char}'(a)$ be the set of characteristic points of $\text{Op}(a)$ in the classical sense (i. e. in the sense of Definition 18.1.25 in [16]). We claim that

$$\text{Char}_{(\omega_0)}(a) \subseteq \text{Char}'(a). \quad (1.7)$$

In fact, assume that $(x_0, \xi_0) \notin \text{Char}'(a)$. This means that there is a neighbourhood X of x_0 , a conical neighbourhood Γ of ξ_0 , $R > 0$ and $b \in S_{1,0}^{-r}(\mathbf{R}^{2d})$ such that $a_r(x, \xi)b(x, \xi) = 1$ when

$$x \in X, \quad \xi \in \Gamma, \quad \text{and} \quad |\xi| > R. \quad (1.8)$$

We shall prove that $a(x, \xi)b(x, \xi) = 1$ when (x, ξ) satisfies (1.8) for some $b \in S_{1,0}^{-r}$ and some choices of X , Γ and R , wherefrom $(x_0, \xi_0) \notin \text{Char}_{(\omega_0)}(a)$.

Since

$$|b(x, \xi)| \leq C \langle \xi \rangle^{-r}, \quad |a_r(x, \xi)| \leq C \langle \xi \rangle^r,$$

and

$$|a(x, \xi) - a_r(x, \xi)| \leq C \langle \xi \rangle^{r-1},$$

for some constant C , it follows that

$$|a_r(x, \xi)| \geq C^{-1} \langle \xi \rangle^r, \quad \text{and} \quad C^{-1} \langle \xi \rangle^r \leq |a(x, \xi)| \leq C \langle \xi \rangle^r,$$

when (x, ξ) satisfies (1.8), for some constants C and R . Hence, if $\varphi \in S_{1,0}^0$ is supported by $X \times \Gamma$, and equal to one in a conical neighborhood of (x_0, ξ_0) , it follows that $b = \varphi \cdot a$ fulfills the required properties. This proves the assertion.

1.3. Modulation spaces. In this subsection we consider properties of modulation spaces which will be used for the definition of wave-front sets of such spaces in Sections 5, and for the proofs of micro-local results for pseudo-differential operators with non-smooth symbols in Section 6. The reader who is not interested in these investigations might immediately pass to Sections 2 and 3.

The short-time Fourier transform of $f \in \mathcal{S}'(\mathbf{R}^d)$ with respect to (the fixed window function) $\phi \in \mathcal{S}(\mathbf{R}^d)$ is defined by

$$(V_\phi f)(x, \xi) = \mathcal{F}(f \cdot \overline{\phi(\cdot - x)})(\xi).$$

We note that the right-hand side makes sense, since it is the partial Fourier transform of the tempered distribution

$$F(x, y) = (f \otimes \bar{\phi})(y, y - x)$$

with respect to the y -variable. If $f \in L^p_{(\omega)}(\mathbf{R}^d)$ for some $p \in [1, \infty]$ and $\omega \in \mathcal{P}(\mathbf{R}^d)$, then $V_\phi f$ takes the form

$$V_\phi f(x, \xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(y) \overline{\phi(y - x)} e^{-i\langle y, \xi \rangle} dy. \quad (1.9)$$

In the following lemma we recall some general continuity properties of the short-time Fourier transform. We omit the proof since the result can be found in e. g. [10, 12].

Lemma 1.5. *Let $f \in \mathcal{S}'(\mathbf{R}^d)$ and $\phi \in \mathcal{S}(\mathbf{R}^d)$. Then the following is true:*

- (1) $V_\phi f \in \mathcal{S}(\mathbf{R}^{2d})$ if and only if $f, \phi \in \mathcal{S}(\mathbf{R}^d)$;
- (2) The map $(f, \phi) \mapsto V_\phi f$ is continuous from $\mathcal{S}(\mathbf{R}^d) \times \mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}(\mathbf{R}^{2d})$, which extends uniquely to a continuous map from $L^2(\mathbf{R}^d) \times L^2(\mathbf{R}^d)$ to $L^2(\mathbf{R}^{2d})$.

If $f \in \mathcal{S}'(\mathbf{R}^d)$ and $\phi \in \mathcal{S}(\mathbf{R}^d)$, then it is well-known that $V_\phi f \in \mathcal{S}' \cap C^\infty$ and

$$|V_\phi f(x, \xi)| \leq C \langle x \rangle^{N_0} \langle \xi \rangle^{N_0},$$

for some constants C and N_0 (see e. g. [12]). If in addition f has compact support, then the following estimate holds (see Proposition 3.6 in [3]).

Lemma 1.6. *Let $f \in \mathcal{E}'(\mathbf{R}^d)$ and $\phi \in \mathcal{S}(\mathbf{R}^d)$. Then for some constant N_0 and every $N \geq 0$, there are constants C_N such that*

$$|V_\phi f(x, \xi)| \leq C_N \langle x \rangle^{-N} \langle \xi \rangle^{N_0}.$$

For further investigations of the short-time Fourier transform, we need the twisted convolution $\widehat{*}$ on $L^1(\mathbf{R}^{2d})$, defined by the formula

$$(F \widehat{*} G)(x, \xi) = (2\pi)^{-d/2} \iint F(x - y, \xi - \eta) G(y, \eta) e^{-i\langle x - y, \eta \rangle} dy d\eta.$$

By straight-forward computations it follows that $\widehat{*}$ restricts to a continuous multiplication on $\mathcal{S}(\mathbf{R}^{2d})$. Furthermore, the map $(F, G) \mapsto F \widehat{*} G$ from $\mathcal{S}(\mathbf{R}^{2d}) \times \mathcal{S}(\mathbf{R}^{2d})$ to $\mathcal{S}(\mathbf{R}^{2d})$ extends uniquely to continuous mappings from $\mathcal{S}'(\mathbf{R}^{2d}) \times \mathcal{S}(\mathbf{R}^{2d})$ and $\mathcal{S}(\mathbf{R}^{2d}) \times \mathcal{S}'(\mathbf{R}^{2d})$ to $\mathcal{S}'(\mathbf{R}^{2d}) \cap C^\infty(\mathbf{R}^{2d})$.

The following lemma is important when proving certain invariance properties for modulation spaces. We omit the proof since the result can be found in [12].

Lemma 1.7. *For each $f \in \mathcal{S}'(\mathbf{R}^d)$ and $\phi_j \in \mathcal{S}(\mathbf{R}^d)$, $j = 1, 2, 3$, it holds*

$$(V_{\phi_1} f) \widehat{*} (V_{\phi_2} \phi_3) = (\phi_3, \phi_1)_{L^2} \cdot V_{\phi_2} f.$$

Now we recall the definition of modulation spaces. Let $\omega \in \mathcal{P}(\mathbf{R}^{2d})$, $p, q \in [1, \infty]$, and $\phi \in \mathcal{S}(\mathbf{R}^d) \setminus 0$ be fixed. Then the *modulation space* $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ is the set of all $f \in \mathcal{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{M_{(\omega)}^{p,q}} = \|f\|_{M_{(\omega)}^{p,q,\phi}} \equiv \|V_\phi f \omega\|_{L_1^{p,q}} < \infty. \quad (1.10)$$

Here $\|\cdot\|_{L_1^{p,q}}$ is the norm given by

$$\|F\|_{L_1^{p,q}} \equiv \left(\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |F(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q},$$

when $F \in L_{loc}^1(\mathbf{R}^{2d})$ (with obvious interpretation when $p = \infty$ or $q = \infty$). Furthermore, the modulation space $W_{(\omega)}^{p,q}(\mathbf{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{W_{(\omega)}^{p,q}} = \|f\|_{W_{(\omega)}^{p,q,\phi}} \equiv \|V_\phi f \omega\|_{L_2^{p,q}} < \infty,$$

where $\|\cdot\|_{L_2^{p,q}}$ is the norm given by

$$\|F\|_{L_2^{p,q}} \equiv \left(\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |F(x, \xi)|^q d\xi \right)^{p/q} dx \right)^{1/p},$$

when $F \in L_{loc}^1(\mathbf{R}^{2d})$.

If $\omega = 1$, then the notation $M^{p,q}$ and $W^{p,q}$ are used instead of $M_{(\omega)}^{p,q}$ and $W_{(\omega)}^{p,q}$ respectively. Moreover we set $M_{(\omega)}^p = W_{(\omega)}^p = M_{(\omega)}^{p,p}$ and $M^p = W^p = M^{p,p}$.

We note that $M^{p,q}$ are modulation spaces of classical form, while $W^{p,q}$ are classical form of Wiener amalgam spaces. We refer to [6] for the most updated definition of modulation spaces.

The following proposition is a consequence of well-known facts in [5] or [12]. Here and in what follows we let p' denote the conjugate exponent of p , i. e. $1/p + 1/p' = 1$.

Proposition 1.8. *Let $p, q, p_j, q_j \in [1, \infty]$ for $j = 1, 2$, and let $\omega, \omega_1, \omega_2, v \in \mathcal{P}(\mathbf{R}^{2d})$. Then the following is true:*

- (1) *if ω is v -moderate and if $\phi \in M_{(v)}^1(\mathbf{R}^d) \setminus 0$, then $f \in M_{(\omega)}^{p,q}(\mathbf{R}^d)$ if and only if (1.10) holds, i. e. $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ is independent of the choice of ϕ . Moreover, $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ is a Banach space under the norm in (1.10), and different choices of ϕ give rise to equivalent norms;*
- (2) *if $p_1 \leq p_2$, $q_1 \leq q_2$ and $\omega_2 \leq C\omega_1$ for some constant C , then*

$$\mathcal{S}(\mathbf{R}^d) \hookrightarrow M_{(\omega_1)}^{p_1, q_1}(\mathbf{R}^d) \hookrightarrow M_{(\omega_2)}^{p_2, q_2}(\mathbf{R}^d) \hookrightarrow \mathcal{S}'(\mathbf{R}^d);$$
- (3) *the sesqui-linear form (\cdot, \cdot) on \mathcal{S} extends to a continuous map from $M_{(\omega)}^{p,q}(\mathbf{R}^d) \times M_{(1/\omega)}^{p',q'}(\mathbf{R}^d)$ to \mathbf{C} . On the other hand, if $\|a\| = \sup |(a, b)|$, where the supremum is taken over all $b \in M_{(1/\omega)}^{p',q'}(\mathbf{R}^d)$ such that $\|b\|_{M_{(1/\omega)}^{p',q'}} \leq 1$, then $\|\cdot\|$ and $\|\cdot\|_{M_{(\omega)}^{p,q}}$ are equivalent norms;*

- (4) if $p, q < \infty$, then $\mathcal{S}(\mathbf{R}^d)$ is dense in $M_{(\omega)}^{p,q}(\mathbf{R}^d)$. The dual space of $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ can be identified with $M_{(1/\omega)}^{p',q'}(\mathbf{R}^d)$, through the form $(\cdot, \cdot)_{L^2}$. Moreover, $\mathcal{S}(\mathbf{R}^d)$ is weakly dense in $M_{(\omega)}^\infty(\mathbf{R}^d)$.

Similar facts hold when the $M_{(\omega)}^{p,q}$ spaces are replaced by $W_{(\omega)}^{p,q}$ spaces.

Proposition 1.8 (1) permits us to be rather vague about the choice of $\phi \in M_{(v)}^1 \setminus 0$ in (1.10). For example, $\|a\|_{M_{(\omega)}^{p,q}} \leq C$ for some $C > 0$ and for every a which belongs to a given subset of \mathcal{S}' , means that the inequality holds for some choice of $\phi \in M_{(v)}^1 \setminus 0$. Evidently, for any other choice of $\phi \in M_{(v)}^1 \setminus 0$, a similar inequality is true although C may have to be replaced by a larger constant, if necessary.

Locally, the spaces $\mathcal{F}L_{(\omega)}^q(\mathbf{R}^d)$, $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ and $W_{(\omega)}^{p,q}(\mathbf{R}^d)$ are the same, in the sense that

$$\mathcal{F}L_{(\omega)}^q(\mathbf{R}^d) \cap \mathcal{E}'(\mathbf{R}^d) = M_{(\omega)}^{p,q}(\mathbf{R}^d) \cap \mathcal{E}'(\mathbf{R}^d) = W_{(\omega)}^{p,q}(\mathbf{R}^d) \cap \mathcal{E}'(\mathbf{R}^d),$$

in view of Remark 4.4 in [22]. In Section 2 and 5 we extend these properties in context of the new type of wave-front sets, and recover the above equalities at the end of Section 4.

Remark 1.9. An example of a space which might be considered as a modulation space and which is neither of the form $M_{(\omega)}^{p,q}$ nor $W_{(\omega)}^{p,q}$ is the space $\widetilde{M}_{(\omega)}$, which consists of all $a \in \mathcal{S}'(\mathbf{R}^{2d})$ such that

$$\|a\|_{\widetilde{M}_{(\omega)}} \equiv \int_{\mathbf{R}^d} \sup_{\zeta \in \mathbf{R}^d} \left(\sup_{x, \xi \in \mathbf{R}^d} |V_\phi a(x, \xi, \zeta, z)| \omega(x, \xi, \zeta, z) \right) dz \quad (1.11)$$

is finite. By straight-forward application of Lemma 1.7 and Young's inequality it follows that

$$M_{(\omega)}^{\infty,1}(\mathbf{R}^{2d}) \subseteq \widetilde{M}_{(\omega)}(\mathbf{R}^{2d}) \subseteq M_{(\omega)}^{\infty,\infty}(\mathbf{R}^{2d}),$$

with continuous embeddings. (Cf. e. g. [6–8, 12].)

1.4. Pseudo-differential operators with non-smooth symbols.

In this subsection we discuss properties of pseudo-differential operators in context of modulation spaces, and start with the following special case of Theorem 4.2 in [29]. We omit the proof.

Proposition 1.10. *Let $p, q, p_j, q_j \in [1, \infty]$ for $j = 1, 2$, be such that*

$$1/p_1 - 1/p_2 = 1/q_1 - 1/q_2 = 1 - 1/p - 1/q, \quad q \leq p_2, q_2 \leq p.$$

Also let $\omega \in \mathcal{P}(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ and $\omega_1, \omega_2 \in \mathcal{P}(\mathbf{R}^{2d})$ be such that

$$\omega(x, \xi, \zeta, z) \leq C \frac{\omega_1(x + z, \xi)}{\omega_2(x, \xi + \zeta)} \quad (1.12)$$

for some constant C . If $a \in M_{(1/\omega)}^{p,q}(\mathbf{R}^{2d})$, then $\text{Op}(a)$ from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$ extends uniquely to a continuous mapping from $M_{(\omega_1)}^{p_1,q_1}(\mathbf{R}^d)$ to $M_{(\omega_2)}^{p_2,q_2}(\mathbf{R}^d)$.

In Section 6 we discuss wave-front set properties for pseudo-differential operators, where the symbol classes are defined by means of modulation spaces. Here, for $\rho \in \mathbf{R}$ and $s \in \mathbf{R}^4$ we define

$$\omega_{s,\rho}(x, \xi, \zeta, z) = \omega(x, \xi, \zeta, z) \langle x \rangle^{-s_4} \langle \zeta \rangle^{-s_3} \langle \xi \rangle^{-\rho s_2} \langle z \rangle^{-s_1}. \quad (1.13)$$

Definition 1.11. Let $\omega \in \mathcal{P}(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$, $s \in \mathbf{R}^4$, $\rho \in \mathbf{R}$, and let $\omega_{s,\rho}$ be as in (1.13). Then the symbol class $\mathcal{U}_{(\omega)}^{s,\rho}(\mathbf{R}^{2d})$ is the set of all $a \in \mathcal{S}'(\mathbf{R}^{2d})$ which satisfy

$$\partial_\xi^\alpha a \in M_{(1/\omega_{s(\alpha),\rho})}^{\infty,1}(\mathbf{R}^{2d}), \quad s(\alpha) = (s_1, |\alpha|, s_3, s_4),$$

for each multi-indices α such that $|\alpha| \leq 2s_2$.

It follows from the following lemma that symbol classes of the form $\mathcal{U}_{(\omega)}^{s,\rho}(\mathbf{R}^{2d})$ are interesting also in the classical theory.

Lemma 1.12. Let $\rho \in [0, 1]$, $\omega \in \mathcal{P}_0(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ and $\omega_0 \in \mathcal{P}_{\rho,0}(\mathbf{R}^{2d})$ be such that

$$\omega_0(x, \xi) = \omega(x, \xi, 0, 0).$$

Then the following conditions are equivalent:

- (1) $a \in S_{\rho,0}^{(\omega_0)}(\mathbf{R}^{2d})$, i. e. (1.6) holds for $\omega = \omega_0$;
- (2) $\omega_0^{-1}a \in S_{\rho,0}^0$;
- (3) $\langle x \rangle^{-s_4}a \in \bigcap_{s_1, s_2, s_3 \geq 0} \mathcal{U}_{(\omega)}^{s,\rho}(\mathbf{R}^{2d})$.

For the proof of Lemma 1.12 we note that

$$\bigcap_{s \geq 0} M_{(v_{r,s})}^{\infty,1}(\mathbf{R}^{2d}) = S_{0,0}^r(\mathbf{R}^{2d}), \quad v_{r,s}(x, \xi, \zeta, z) = \langle \xi \rangle^{-r} \langle \zeta \rangle^s \langle z \rangle^s, \quad (1.14)$$

which follows from Theorem 2.2 in [28] (see also Remark 2.18 in [15]).

Proof. In order to prove the equivalence between (1) and (2) we note that the condition $\omega_0 \in \mathcal{P}_{\rho,0}$ implies that $\omega_0 \in S_{\rho,0}^{(\omega_0)}(\mathbf{R}^{2d})$ and $\omega_0^{-1} \in S_{\rho,0}^{(1/\omega_0)}(\mathbf{R}^{2d})$. Hence, if $a \in S_{\rho,0}^{(\omega_0)}(\mathbf{R}^{2d})$, then it follows by straightforward computations that

$$\omega_0^{-1}a \in S_{\rho,0}^{(1/\omega_0)} \cdot S_{\rho,0}^{(\omega_0)} = S_{\rho,0}^{(\omega_0^{-1}\omega_0)} = S_{\rho,0}^{(1)} = S_{\rho,0}^0$$

(see also Lemma 18.4.3 in [16]). This proves that (1) implies (2), and in the same way the opposite implication follows.

Next we consider (3). We observe that for some positive constants C and N we have

$$C^{-1}\omega_0(x, \xi) \langle \zeta \rangle^{-N} \langle z \rangle^{-N} \leq \omega(x, \xi, \zeta, z) \leq C\omega_0(x, \xi) \langle \zeta \rangle^N \langle z \rangle^N,$$

which implies that

$$\bigcap_{s_1, s_2, s_3 \geq 0} \mathcal{U}_{(\omega)}^{s,\rho}(\mathbf{R}^{2d}) = \bigcap_{s_1, s_2, s_3 \geq 0} \mathcal{U}_{(\omega_0)}^{s,\rho}(\mathbf{R}^{2d}).$$

Since the map $a \mapsto \omega_0^{-1}a$ is a homeomorphism from $M_{(\omega_1/\omega_0)}^{\infty,1}$ to $M_{(\omega_1)}^{\infty,1}$ when $\omega_1 \in \mathcal{P}$, by Theorem 2.2 in [28], we may assume that $\omega =$

$\omega_0 = 1$. Furthermore we may assume that $s_4 = 0$, since (3) is invariant under the choice of s_4 . For such choices of parameters, the asserted equivalences can be formulated as

$$\bigcap_{s_1, s_2, s_3 \geq 0} \mathcal{U}_{(\omega)}^{s, \rho}(\mathbf{R}^{2d}) = S_{\rho, 0}^0(\mathbf{R}^{2d}), \quad \omega = 1.$$

The result is now an immediate consequence of (1.14) and the fact that $a \in S_{\rho, 0}^0(\mathbf{R}^{2d})$, if and only if for each multi-indices α and β , it holds

$$\partial_x^\alpha \partial_\xi^\beta a \in S_{0, 0}^{-\rho|\beta|}(\mathbf{R}^{2d}).$$

The proof is complete. \square

2. WAVE FRONT SETS WITH RESPECT TO FOURIER LEBESGUE SPACES

In this section we define wave-front sets with respect to Fourier Lebesgue spaces, and show some basic properties.

Assume that $\omega \in \mathcal{P}(\mathbf{R}^{2d})$, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ is an open cone and $q \in [1, \infty]$ are fixed. For any $f \in \mathcal{S}'(\mathbf{R}^d)$, let

$$|f|_{\mathcal{F}L_{(\omega)}^{q, \Gamma}} = |f|_{\mathcal{F}L_{(\omega), x}^{q, \Gamma}} \equiv \left(\int_{\Gamma} |\widehat{f}(\xi) \omega(x, \xi)|^q d\xi \right)^{1/q} \quad (2.1)$$

(with obvious interpretation when $q = \infty$). We note that $|\cdot|_{\mathcal{F}L_{(\omega), x}^{q, \Gamma}}$ defines a semi-norm on \mathcal{S}' which might attain the value $+\infty$. Since ω is v -moderate for some $v \in \mathcal{P}(\mathbf{R}^{2d})$, it follows that different $x \in \mathbf{R}^d$ gives rise to equivalent semi-norms $|f|_{\mathcal{F}L_{(\omega), x}^{q, \Gamma}}$. Furthermore, if $\Gamma = \mathbf{R}^d \setminus 0$, $f \in \mathcal{F}L_{(\omega)}^q(\mathbf{R}^d)$ and $q < \infty$, then $|f|_{\mathcal{F}L_{(\omega), x}^{q, \Gamma}}$ agrees with the Fourier Lebesgue norm $\|f\|_{\mathcal{F}L_{(\omega), x}^q}$ of f .

For the sake of notational convenience we set

$$\mathcal{B} = \mathcal{F}L_{(\omega)}^q = \mathcal{F}L_{(\omega)}^q(\mathbf{R}^d), \quad \text{and} \quad |\cdot|_{\mathcal{B}(\Gamma)} = |\cdot|_{\mathcal{F}L_{(\omega), x}^{q, \Gamma}}. \quad (2.2)$$

We let $\Theta_{\mathcal{B}}(f) = \Theta_{\mathcal{F}L_{(\omega)}}(f)$ be the set of all $\xi \in \mathbf{R}^d \setminus 0$ such that $|f|_{\mathcal{B}(\Gamma)} < \infty$, for some $\Gamma = \Gamma_\xi$. We also let $\Sigma_{\mathcal{B}}(f)$ be the complement of $\Theta_{\mathcal{B}}(f)$ in $\mathbf{R}^d \setminus 0$. Then $\Theta_{\mathcal{B}}(f)$ and $\Sigma_{\mathcal{B}}(f)$ are open respectively closed subsets in $\mathbf{R}^d \setminus 0$, which are independent of the choice of $x \in \mathbf{R}^d$ in (2.1).

Definition 2.1. Let $q \in [1, \infty]$, $\omega \in \mathcal{P}(\mathbf{R}^{2d})$, \mathcal{B} be as in (2.2), and let X be an open subset of \mathbf{R}^d . The wave-front set of $f \in \mathcal{D}'(X)$, $\text{WF}_{\mathcal{B}}(f) \equiv \text{WF}_{\mathcal{F}L_{(\omega)}}(f)$ with respect to \mathcal{B} consists of all pairs (x_0, ξ_0) in $X \times (\mathbf{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_{\mathcal{B}}(\varphi f)$ holds for each $\varphi \in C_0^\infty(X)$ such that $\varphi(x_0) \neq 0$.

We note that $\text{WF}_{\mathcal{B}}(f)$ is a closed set in $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, since it is obvious that its complement is open. We also note that if $x \in \mathbf{R}^d$ is

fixed and $\omega_0(\xi) = \omega(x, \xi)$, then $\text{WF}_{\mathcal{B}}(f) = \text{WF}_{\mathcal{F}L_{(\omega_0)}^q}(f)$, since $\Sigma_{\mathcal{B}}$ is independent of x .

The following theorem shows that wave-front sets with respect to $\mathcal{F}L_{(\omega)}^q$ satisfy appropriate micro-local properties. It also shows that such wave-front sets are decreasing with respect to the parameter q , and increasing with respect to the weight ω .

Theorem 2.2. *Let $X \subseteq \mathbf{R}^d$ be open, $q, r \in [1, \infty]$ and $\omega, \vartheta \in \mathcal{P}(\mathbf{R}^{2d})$ be such that*

$$q \leq r, \quad \text{and} \quad \vartheta(x, \xi) \leq C\omega(x, \xi), \quad (2.3)$$

for some constant C which is independent of $x, \xi \in \mathbf{R}^d$. Also let \mathcal{B} be as in (2.2) and put $\mathcal{B}_0 = \mathcal{F}L_{(\vartheta)}^r = \mathcal{F}L_{(\vartheta)}^r(\mathbf{R}^d)$. If $f \in \mathcal{D}'(X)$ and $\varphi \in C^\infty(X)$, then $\text{WF}_{\mathcal{B}_0}(\varphi f) \subseteq \text{WF}_{\mathcal{B}}(f)$.

Proof. It suffices to prove

$$\Sigma_{\mathcal{B}_0}(\varphi f) \subseteq \Sigma_{\mathcal{B}}(f). \quad (2.4)$$

when $\varphi \in \mathcal{S}(\mathbf{R}^d)$ and $f \in \mathcal{E}'(\mathbf{R}^d)$, since the statement only involve local assertions. For the same reasons we may assume that $\omega(x, \xi) = \omega(\xi)$ is independent of x . It is also no restrictions to assume that $\vartheta = \omega$.

Let $\xi_0 \in \Theta_{\mathcal{B}}(f)$, and choose open cones Γ_1 and Γ_2 in \mathbf{R}^d such that $\overline{\Gamma_2} \subseteq \Gamma_1$. Since f has compact support, it follows that $|\widehat{f}(\xi)\omega(\xi)| \leq C\langle \xi \rangle^{N_0}$ for some positive constants C and N_0 . The result therefore follows if we prove that for each N , there are constants C_N such that

$$|\varphi f|_{\mathcal{B}_0(\Gamma_2)} \leq C_N \left(|f|_{\mathcal{B}(\Gamma_1)} + \sup_{\xi \in \mathbf{R}^d} (|\widehat{f}(\xi)\omega(\xi)|\langle \xi \rangle^{-N}) \right) \quad (2.5)$$

when $\overline{\Gamma_2} \subseteq \Gamma_1$ and $N = 1, 2, \dots$

By using the fact that ω is v -moderate for some $v \in \mathcal{P}(\mathbf{R}^d)$, and letting $F(\xi) = |\widehat{f}(\xi)\omega(\xi)|$ and $\psi(\xi) = |\widehat{\varphi}(\xi)v(\xi)|$, it follows that ψ turns rapidly to zero at infinity and

$$\begin{aligned} |\varphi f|_{\mathcal{B}_0(\Gamma_2)} &= \left(\int_{\Gamma_2} |\mathcal{F}(\varphi f)(\xi)\omega(\xi)|^r d\xi \right)^{1/r} \\ &\leq C \left(\int_{\Gamma_2} \left(\int_{\mathbf{R}^d} \psi(\xi - \eta) F(\eta) d\eta \right)^r d\xi \right)^{1/r} \leq C(J_1 + J_2) \end{aligned}$$

for some constant C , where

$$J_1 = \left(\int_{\Gamma_2} \left(\int_{\Gamma_1} \psi(\xi - \eta) F(\eta) d\eta \right)^r d\xi \right)^{1/r}$$

and

$$J_2 = \left(\int_{\Gamma_2} \left(\int_{\mathbb{G}\Gamma_1} \psi(\xi - \eta) F(\eta) d\eta \right)^r d\xi \right)^{1/r}.$$

Let q_0 be chosen such that $1/q_0 + 1/q = 1 + 1/r$, and let χ_{Γ_1} be the characteristic function of Γ_1 . Then Young's inequality gives

$$\begin{aligned} J_1 &\leq \left(\int_{\mathbf{R}^d} \left(\int_{\Gamma_1} \psi(\xi - \eta) F(\eta) d\eta \right)^r d\xi \right)^{1/r} \\ &= \|\psi * (\chi_{\Gamma_1} F)\|_{L^r} \leq \|\psi\|_{L^{q_0}} \|\chi_{\Gamma_1} F\|_{L^q} = C_\psi |f|_{\mathcal{B}(\Gamma_1)}, \end{aligned}$$

where $C_\psi = \|\psi\|_{L^{q_0}} < \infty$ since ψ turns rapidly to zero at infinity.

In order to estimate J_2 , we note that the conditions $\xi \in \Gamma_2$, $\eta \notin \Gamma_1$ and the fact that $\overline{\Gamma_2} \subseteq \Gamma_1$ imply that $|\xi - \eta| > c \max(|\xi|, |\eta|)$ for some constant $c > 0$, since this is true when $1 = |\xi| \geq |\eta|$. Since ψ turns rapidly to zero at infinity, it follows that for each $N_0 > d$ and $N \in \mathbf{N}$ such that $N > N_0$, we have

$$\begin{aligned} J_2 &\leq C_1 \left(\int_{\Gamma_2} \left(\int_{\mathbb{C}\Gamma_1} \langle \xi - \eta \rangle^{-(2N_0+N)} F(\eta) d\eta \right)^r d\xi \right)^{1/r} \\ &\leq C_2 \left(\int_{\Gamma_2} \left(\int_{\mathbb{C}\Gamma_1} \langle \xi \rangle^{-N_0} \langle \eta \rangle^{-N_0} (\langle \eta \rangle^{-N} F(\eta)) d\eta \right)^r d\xi \right)^{1/r} \\ &\leq C_3 \sup_{\eta \in \mathbf{R}^d} |\langle \eta \rangle^{-N} F(\eta)|, \end{aligned}$$

for some constants $C_1, C_2, C_3 > 0$. This proves (2.5). The proof is complete. \square

3. WAVE-FRONT SETS FOR PSEUDO-DIFFERENTIAL OPERATORS WITH SMOOTH SYMBOLS

In this section we establish mapping properties for pseudo-differential operators on wave-front sets of Fourier Lebesgue types. More precisely, we prove the following result (cf. (*) in the abstract).

Theorem 3.1. *Let $\rho > 0$, $\omega \in \mathcal{P}(\mathbf{R}^{2d})$, $\omega_0 \in \mathcal{P}_{\rho,0}(\mathbf{R}^{2d})$, $a \in S_{\rho,0}^{(\omega_0)}(\mathbf{R}^{2d})$, $f \in \mathcal{S}'(\mathbf{R}^d)$ and $q \in [1, \infty]$. Also let*

$$\mathcal{B} = \mathcal{F}L_{(\omega)}^q = \mathcal{F}L_{(\omega)}^q(\mathbf{R}^d) \quad \text{and} \quad \mathcal{C} = \mathcal{F}L_{(\omega/\omega_0)}^q = \mathcal{F}L_{(\omega/\omega_0)}^q(\mathbf{R}^d).$$

Then

$$\text{WF}_{\mathcal{C}}(\text{Op}(a)f) \subseteq \text{WF}_{\mathcal{B}}(f) \subseteq \text{WF}_{\mathcal{C}}(\text{Op}(a)f) \cup \text{Char}_{(\omega)}(a). \quad (3.1)$$

We need some preparations for the proof. The first proposition shows that if $x_0 \notin \text{supp } f$ then $(x_0, \xi) \notin \text{WF}_{\mathcal{C}}(\text{Op}(a)f)$ for every $\xi \in \mathbf{R}^d \setminus 0$.

Proposition 3.2. *Let $\omega \in \mathcal{P}(\mathbf{R}^{2d})$, $\omega_0 \in \mathcal{P}_{\rho,\delta}(\mathbf{R}^{2d})$, $0 \leq \delta \leq \rho$, $0 < \rho$, $\delta < 1$, and let $a \in S_{\rho,\delta}^{(\omega_0)}(\mathbf{R}^{2d})$. Also let \mathcal{C} be as in Theorem 3.1 and let the operator L_a on $\mathcal{S}'(\mathbf{R}^d)$ be defined by the formula*

$$(L_a f)(x) \equiv \varphi_1(x)(\text{Op}(a)(\varphi_2 f))(x), \quad f \in \mathcal{S}'(\mathbf{R}^d), \quad (3.2)$$

where $\varphi_1 \in C_0^\infty(\mathbf{R}^d)$ and $\varphi_2 \in S_{0,0}^0(\mathbf{R}^d)$ are such that

$$\text{supp } \varphi_1 \cap \text{supp } \varphi_2 = \emptyset.$$

Then the kernel of L_a belongs to $\mathcal{S}(\mathbf{R}^{2d})$. In particular, the following is true:

- (1) $L_a = \text{Op}(a_0)$ for some $a_0 \in \mathcal{S}(\mathbf{R}^{2d})$;
- (2) $\text{WF}_C(L_a f) = \emptyset$, for any given $q \in [1, \infty]$.

Proof. We note that a_0 exists as a tempered distribution in view of Section 1. We need to prove that $a_0 \in \mathcal{S}$, or equivalently, that the kernel K_a of L_a belongs to \mathcal{S} . By the definition it follows that

$$(L_a f)(x) = (2\pi)^{-d} \iint a(x, \xi) \varphi_1(x) \varphi_2(y) f(y) e^{i\langle x-y, \xi \rangle} dy d\xi.$$

Since φ_1 has compact support it follows that for some $\varepsilon > 0$ it holds $\varphi_1(x) \varphi_2(y) = 0$ when $|x - y| \leq 2\varepsilon$. Hence, if $\varphi \in C^\infty(\mathbf{R}^d)$ satisfy $\varphi(x) = 0$ when $|x| \leq \varepsilon$ and $\varphi(x) = 1$ when $|x| \geq 2\varepsilon$, $f_2 = \varphi_2 f$ and $a_1 = \varphi_1 a$, then it follows by partial integrations that

$$\begin{aligned} (L_a f)(x) &= \text{Op}(a_1) f_2(x) = (2\pi)^{-d} \iint a_1(x, \xi) f_2(y) e^{i\langle x-y, \xi \rangle} dy d\xi \\ &= (2\pi)^{-d} \iint (-1)^{s_2} (\Delta_\xi^{s_2} a_1)(x, \xi) f_2(y) |x - y|^{-2s_2} e^{i\langle x-y, \xi \rangle} dy d\xi \\ &= (2\pi)^{-d} \iint (-1)^{s_2} (\Delta_\xi^{s_2} a_1)(x, \xi) f_2(y) \varphi(x - y) |x - y|^{-2s_2} e^{i\langle x-y, \xi \rangle} dy d\xi \\ &= (\text{Op}(b_s) f)(x), \end{aligned}$$

where $s_2 = s \geq 0$ is an integer,

$$b_s(x, y, \xi) = (-1)^{s_2} (\Delta_\xi^{s_2} a_1)(x, \xi) \varphi_1(x) \varphi_2(y) \varphi(x - y) |x - y|^{-2s_2} \quad (3.3)$$

and

$$\text{Op}(b_s) f(x) = (2\pi)^{-d} \iint b_s(x, y, \xi) f(y) e^{i\langle x-y, \xi \rangle} dy d\xi.$$

>From the fact that $|x - y| \geq C\langle x - y \rangle$, when $(x, y, \xi) \in \text{supp } b_s$, and that $a \in S_{\rho, \delta}^{(\omega_0)}(\mathbf{R}^{2d})$, it follows from (3.3) that

$$\begin{aligned} |b_s(x, y, \xi)| &\leq C_s \omega_0(x, \xi) \langle x \rangle^{-2s} \langle y \rangle^{-2s} \langle \xi \rangle^{-2\rho s} \\ &\leq C'_s \langle x \rangle^{N_0 - 2s} \langle y \rangle^{-2s} \langle \xi \rangle^{N_0 - 2\rho s}, \end{aligned}$$

for some constant N_0 which is independent of s . In the same way it follows that

$$|\partial^\alpha b_s(x, y, \xi)| \leq C_{s, \alpha} \langle x \rangle^{N_0 - 2s} \langle y \rangle^{-2s} \langle \xi \rangle^{N_0 - 2\rho s}, \quad (3.4)$$

for some constant N_0 which depends on α , but is independent of s .

Now let $N \geq 0$ be arbitrary. Since the distribution kernel K_a of L_a is equal to

$$(2\pi)^{-d} \int b_s(x, y, \xi) e^{i\langle x-y, \xi \rangle} d\xi,$$

it follows by choosing s large enough in (3.4) that for each multi-index α , there is a constant $C_{\alpha, N}$ such that

$$|\partial^\alpha K_a(x, y)| \leq C_{\alpha, N} \langle x, y \rangle^{-N}.$$

This proves that $K_a \in \mathcal{S}(\mathbf{R}^{2d})$, and (1) follows.

The assertion (2) is an immediate consequence of (1). The proof is complete. \square

Next we consider properties of the wave-front set of $\text{Op}(a)f$ at a fixed point when f is concentrated to that point.

Proposition 3.3. *Let $\rho, \omega, \omega_0, a, \mathcal{B}$ and \mathcal{C} be as in Theorem 3.1. Also let $q \in [1, \infty]$ and $f \in \mathcal{E}'(\mathbf{R}^d)$. Then the following is true:*

- (1) *if Γ_1, Γ_2 are open cones in $\subseteq \mathbf{R}^d \setminus 0$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$, and $|f|_{\mathcal{B}(\Gamma_1)} < \infty$, then $|\text{Op}(a)f|_{\mathcal{C}(\Gamma_2)} < \infty$;*
- (2) $\text{WF}_{\mathcal{C}}(\text{Op}(a)f) \subseteq \text{WF}_{\mathcal{B}}(f)$.

We note that $\text{Op}(a)f$ in Proposition 3.3 makes sense as an element in $\mathcal{S}'(\mathbf{R}^d)$, by Proposition 1.10.

Proof. We may assume that $\omega(x, \xi) = \omega(\xi)$, $\omega_0(x, \xi) = \omega_0(\xi)$, and that $\text{supp } a \subseteq K \times \mathbf{R}^d$ for some compact set $K \subseteq \mathbf{R}^d$, since the statements only involve local assertions. We only prove the result for $q < \infty$. The slight modifications to the case $q = \infty$ are left for the reader.

By straight-forward computations we get

$$\mathcal{F}(\text{Op}(a)f)(\xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} (\mathcal{F}_1 a)(\xi - \eta, \eta) \widehat{f}(\eta) d\eta, \quad (3.5)$$

where $\mathcal{F}_1 a$ denotes the partial Fourier transform of $a(x, \xi)$ with respect to the x -variable. We need to estimate the modulus of $\mathcal{F}_1 a(\eta, \xi)$.

From the fact that $a \in S_{\rho, 0}^{(\omega_0)}(\mathbf{R}^{2d})$ is smooth and compactly supported in the x variable, it follows that for each $N \geq 0$, there is a constant C_N such that

$$|(\mathcal{F}_1 a)(\xi, \eta)| \leq C_N \langle \xi \rangle^{-N} \omega_0(\eta). \quad (3.6)$$

Hence the facts that $\omega(\eta) \leq \omega(\xi) \langle \xi - \eta \rangle^{N_0}$ and $\omega_0(\eta) \leq \omega_0(\xi) \langle \xi - \eta \rangle^{N_0}$ for some N_0 give that for each $N > d$ it holds

$$\begin{aligned} |(\mathcal{F}_1 a)(\xi - \eta, \eta) \omega(\xi) / \omega_0(\xi)| &\leq C_N \langle \xi - \eta \rangle^{-(N+2N_0)} \omega_0(\eta) \omega(\xi) / \omega_0(\xi) \\ &\leq C'_N \langle \xi - \eta \rangle^{-N} \omega(\eta), \end{aligned} \quad (3.7)$$

for some constants C_N and C'_N .

By letting $F(\xi) = |\widehat{f}(\xi)\omega(\xi)|$, then (3.5), (3.7) and Hölder's inequality give

$$\begin{aligned} |\mathcal{F}(\text{Op}(a)f)(\xi)\omega_2(\xi)| &\leq C \int_{\mathbf{R}^d} \langle \xi - \eta \rangle^{-N} F(\eta) d\eta, \\ &= C \int_{\mathbf{R}^d} (\langle \xi - \eta \rangle^{-N/q} F(\eta)) \langle \xi - \eta \rangle^{-N/q'} d\eta \\ &\leq C' \left(\int_{\mathbf{R}^d} \langle \xi - \eta \rangle^{-N} F(\eta)^q d\eta \right)^{1/q}, \end{aligned} \quad (3.8)$$

where $C = (2\pi)^{-d/2} C_N''$ and

$$C' = C \|\langle \cdot \rangle^{-N}\|_{L^1}^{1/q'}. \quad (3.9)$$

Here $C' < \infty$ in (3.9), since $N > d$.

We have to estimate

$$|(\text{Op}(a)f)|_{C(\Gamma_2)} = \left(\int_{\Gamma_2} |\mathcal{F}(\text{Op}(a)f)(\xi)\omega(\xi)/\omega_0(\xi)|^q d\xi \right)^{1/q}.$$

By (3.8) we get

$$\begin{aligned} &\left(\int_{\Gamma_2} |\mathcal{F}(\text{Op}(a)f)(\xi)\omega(\xi)/\omega_0(\xi)|^q d\xi \right)^{1/q} \\ &\leq C \left(\iint_{\xi \in \Gamma_2} \langle \xi - \eta \rangle^{-N} F(\eta)^q d\eta d\xi \right)^{1/q} \leq C(J_1 + J_2), \end{aligned}$$

for some constant C , where

$$J_1 = \left(\int_{\Gamma_2} \int_{\Gamma_1} \langle \xi - \eta \rangle^{-N} F(\eta)^q d\eta d\xi \right)^{1/q}$$

and

$$J_2 = \left(\int_{\Gamma_2} \int_{\mathbb{C}\Gamma_1} \langle \xi - \eta \rangle^{-N} F(\eta)^q d\eta d\xi \right)^{1/q}.$$

In order to estimate J_1 and J_2 we argue as in the proof of (2.5). More precisely, for J_1 we have

$$\begin{aligned} J_1 &\leq \left(\int_{\mathbf{R}^d} \int_{\Gamma_1} \langle \xi - \eta \rangle^{-N} F(\eta)^q d\eta d\xi \right)^{1/q} \\ &= \left(\int_{\mathbf{R}^d} \int_{\Gamma_1} \langle \xi \rangle^{-N} F(\eta)^q d\eta d\xi \right)^{1/q} = C \left(\int_{\Gamma_1} F(\eta)^q d\eta \right)^{1/q} < \infty. \end{aligned}$$

In order to estimate J_2 , we assume from now on that Γ_2 is chosen such that $\Gamma_2 \subseteq \Gamma_1$, and that the distance between the boundaries of Γ_1 and Γ_2 on the $d-1$ dimensional unit sphere \mathbf{S}^{d-1} is larger than $r > 0$. This gives

$$|\xi - \eta| > r \quad \text{when} \quad \xi \in \Gamma_2 \cap \mathbf{S}^{d-1}, \text{ and } \eta \in (\mathbb{C}\Gamma_1) \cap \mathbf{S}^{d-1}. \quad (3.10)$$

Then for some constant $c > 0$ we have

$$|\xi - \eta| \geq c \max(|\xi|, |\eta|), \quad \text{when } \xi \in \Gamma_2, \text{ and } \eta \in \mathbb{C}\Gamma_1.$$

In fact, when proving this we may assume that $|\eta| \leq |\xi| = 1$. Then we must have that $|\xi - \eta| \geq c$ for some constant $c > 0$, since we otherwise get a contradiction of (3.10).

Since f has compact support, it follows that $F(\eta) \leq C\langle\eta\rangle^{t_1}$ for some constant C . By combining these estimates we obtain

$$\begin{aligned} J_2 &\leq \left(\int_{\Gamma_2} \int_{\mathbb{C}\Gamma_1} F(\eta) \langle \xi - \eta \rangle^{-N} d\eta d\xi \right)^{1/q} \\ &\leq C \left(\int_{\Gamma_2} \int_{\mathbb{C}\Gamma_1} \langle \eta \rangle^{t_1} \langle \xi \rangle^{-N/2} \langle \eta \rangle^{-N/2} d\eta d\xi \right). \end{aligned}$$

Hence, if we choose $N > 2d + 2t_1$, it follows that the right-hand side is finite. This proves (1).

The assertion (2) follows immediately from (1) and the definitions. The proof is complete. \square

We also need the following lemma. Here we recall Definition 1.3 for notations.

Lemma 3.4. *Let $0 \leq \delta < \rho \leq 1$, $\mu = \rho - \delta$, $\omega_0 \in \mathcal{P}_{\rho,\delta}(\mathbf{R}^{2d})$, and $a \in S_{\rho,\delta}^{(\omega_0)}(\mathbf{R}^{2d})$. If $(x_0, \xi_0) \notin \text{Char}_{(\omega_0)}(a)$, then for some open cone $\Gamma = \Gamma_{\xi_0}$, open neighborhood $X \subseteq \mathbf{R}^d$ of x_0 and $R > 0$, there are elements $c_j \in S_{\rho,\delta}^0$ which are (X, Γ, R) -unitary, $b_j \in S_{\rho,\delta}^{(1/\omega_0)}$ and $h_j \in S_{\rho,\delta}^{-j\mu}$ for $j \in \mathbf{N}$ such that*

$$\text{Op}(b_j) \text{Op}(a) = \text{Op}(c_j) + \text{Op}(h_j), \quad j \geq 1.$$

Proof. For $j = 1$, the result is obvious in view of Definition 1.3. Therefore assume that $j > 1$, and that b_k, c_k and h_k for $k = 1, \dots, j-1$ have already been chosen which satisfy the required properties. Then we inductively define b_j by the formula

$$\text{Op}(b_j) = (\text{Op}(c_{j-1}) - \text{Op}(h_{j-1})) \text{Op}(b_{j-1}).$$

By the inductive hypothesis it follows that

$$\text{Op}(b_j) \text{Op}(a) = \text{Op}(\tilde{c}) + \text{Op}(\tilde{h}_1) + \text{Op}(\tilde{h}_2),$$

where

$$\text{Op}(\tilde{c}) = \text{Op}(c_{j-1}) \text{Op}(c_{j-1}),$$

$$\text{Op}(\tilde{h}_1) = [\text{Op}(c_{j-1}), \text{Op}(h_{j-1})]$$

and

$$\text{Op}(\tilde{h}_2) = \text{Op}(h_{j-1}) \text{Op}(h_{j-1}).$$

Here $[\cdot, \cdot]$ denotes the commutator between operators.

By Theorems 18.5.4 and 18.5.10 in [16] it follows that $\tilde{h}_2 \in S_{\rho,\delta}^{-2(j-1)\mu} \subseteq S_{\rho,\delta}^{-j\mu}$, since the conditions $j-1 \geq 1$ and $0 < \mu \leq 1$ imply that $-2(j-1)\mu \leq -j\mu$. Hence

$$\tilde{h}_l \in S_{\rho,\delta}^{-j\mu} \quad (3.11)$$

holds for $l = 2$.

Next we consider the term $\text{Op}(\tilde{h}_1)$. By Theorem 18.1.18 [16] it follows that

$$\begin{aligned} \tilde{h}_1(x, \xi) = i \sum_{|\alpha|=1} (\partial_x^\alpha c_{j-1}(x, \xi) \partial_\xi^\alpha h_{j-1}(x, \xi) - \partial_\xi^\alpha c_{j-1}(x, \xi) \partial_x^\alpha h_{j-1}(x, \xi)) \\ + \tilde{h}_3(x, \xi), \end{aligned}$$

for some $\tilde{h}_3 \in S_{\rho,\delta}^{-(j+1)\mu}$. Since the sum belongs to $S_{\rho,\delta}^{-j\mu}$ in view of the definitions, it follows that (3.11) also holds for $l = 1$.

It remains to consider the term \tilde{c} . By Theorems 18.5.4 and 18.5.10 in [16] again, it follows that

$$\tilde{c}(x, \xi) = c_j(x, \xi) + \tilde{h}_4(x, \xi),$$

where

$$c_j(x, \xi) = \sum_{|\alpha| \leq N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_x^\alpha c_{j-1}(x, \xi) \partial_\xi^\alpha c_{j-1}(x, \xi),$$

and $\tilde{h}_4 \in S_{\rho,\delta}^{-j\mu}$, provided N was chosen sufficiently large. Since $c_{j-1} = 1$ on

$$\{(x, \xi); x \in X, \xi \in \Gamma, |\xi| > R\},$$

it follows that c_j is (X, Γ, R) -unitary. The result now follows by letting $h_j = \tilde{h}_1 + \tilde{h}_2 + \tilde{h}_4$. The proof is complete. \square

Proof of Theorem 3.1. We start to prove the first inclusion in (3.1). Assume that $(x_0, \xi_0) \notin \text{WF}_B(f)$, let $\varphi \in C_0^\infty(\mathbf{R}^d)$ be such that $\varphi = 1$ in a neighborhood of x_0 , and set $\varphi_1 = 1 - \varphi$. Then it follows from Proposition 3.2 that

$$(x_0, \xi_0) \notin \text{WF}_C(\text{Op}(a)(\varphi_1 f)).$$

Furthermore, by Proposition 3.3 we get

$$(x_0, \xi_0) \notin \text{WF}_C(\text{Op}(a)(\varphi f)),$$

since if $a_0(x, \xi) = \varphi(x)a(x, \xi)$, then $\text{Op}(a)(\varphi f)$ is equal to $\text{Op}(a_0)(\varphi f)$ near x_0 . The first embedding in (3.1) is now a consequence of the inclusion

$$\text{WF}_C(\text{Op}(a)f) \subseteq \text{WF}_C(\text{Op}(a)(\varphi f)) \cup \text{WF}_C(\text{Op}(a)(\varphi_1 f)).$$

It remains to prove the last inclusion in (3.1). By Proposition 3.2 it follows that it is no restriction to assume that f has compact support. Assume that

$$(x_0, \xi_0) \notin \text{WF}_C(\text{Op}(a)f) \cup \text{Char}_{(\omega_0)}(a),$$

and choose b_j, c_j and h_j as in Lemma 3.4. We shall prove that $(x_0, \xi_0) \notin \text{WF}_B(f)$. Since

$$f = \text{Op}(1 - c_j)f + \text{Op}(b_j) \text{Op}(a)f + \text{Op}(h_j)f,$$

the result follows if we prove

$$(x_0, \xi_0) \notin S_1 \cup S_2 \cup S_3,$$

where

$$S_1 = \text{WF}_B(\text{Op}(1 - c_j)f), \quad S_2 = \text{WF}_B(\text{Op}(b_j) \text{Op}(a)f)$$

$$\text{and } S_3 = \text{WF}_B(\text{Op}(h_j)f).$$

We start to consider S_2 . By the first embedding in (3.1) it follows that

$$S_2 = \text{WF}_B(\text{Op}(b_j) \text{Op}(a)f) \subseteq \text{WF}_C(\text{Op}(a)f).$$

Since we have assumed that $(x_0, \xi_0) \notin \text{WF}_C(\text{Op}(a)f)$, it follows that $(x_0, \xi_0) \notin S_2$.

Next we consider S_3 . Since f has compact support and $\omega, \omega_0 \in \mathcal{P}(\mathbf{R}^{2d})$, it follows from Lemma 3.4 that for each $N \geq 0$, there is a $j \geq 1$ such that $\varphi(x)D^\alpha(\text{Op}(h_j)f) \in L^\infty$ when $|\alpha| \leq N$ and $\varphi \in C_0^\infty(\mathbf{R}^d)$ for some $h_j \in S_{\rho,0}^{-j\rho}$. This implies that S_3 is empty, provided N (and therefore j) was chosen large enough.

Finally we consider S_1 . By the assumptions it follows that $a_0 = 1 - c_j = 0$ in Γ , and by replacing Γ with a smaller cone, if necessary, we may assume that $a_0 = 0$ in a conical neighborhood of Γ . Hence, if $\Gamma = \Gamma_1$, and Γ_1, Γ_2, J_1 and J_2 are the same as in the proof of Proposition 3.3, then it follows from that proof and the fact that $a_0(x, \xi) \in S_{\rho,0}^0$ is compactly supported in the x -variable, that $J_1 < +\infty$ and for each $N \geq 0$, there are constants C_N and C'_N such that

$$\begin{aligned} |\text{Op}(a_0)f|_{B(\Gamma_2)} &\leq C_N(J_1 + J_2) \\ &\leq C'_N \left(J_1 + \left(\int_{\Gamma_2} \int_{\mathbf{R}^d} \langle \xi \rangle^{-N/2} \langle \eta \rangle^{(N_0 - N/2)} d\eta d\xi \right)^{1/q} \right). \end{aligned} \quad (3.12)$$

for some $N_0 \geq 0$. By choosing $N > 2N_0 + 2d$, it follows that $|\text{Op}(a_0)f|_{C(\Gamma_2)} < \infty$. This proves that $(x_0, \xi_0) \notin S_1$, and the proof is complete. \square

Remark 3.5. By Theorem 18.5.10 in [16] it follows that the first embedding in (3.1) remains valid if $\text{Op}(a)$ is replaced by $\text{Op}_t(a)$.

Remark 3.6. We note that the inclusions in Theorem 3.1 may be violated when $\omega_0 = 1$ and the assumption $\rho > 0$ is replaced by $\rho = 0$. In fact, let $a(x, \xi) = e^{-i\langle x_0, \xi \rangle}$ for some fixed $x_0 \in \mathbf{R}^d$, and choose α in such way that $f_\alpha(x) = \delta_0^{(\alpha)}$ does not belong to $\mathcal{B} = \mathcal{F}L_{(\omega)}^q(\mathbf{R}^d)$. Since

$$(\text{Op}(a)f_\alpha)(x) = f_\alpha(x - x_0),$$

by straight-forward computations, it follows that for some closed cone Γ in $\mathbf{R}^d \setminus 0$ we have

$$\text{WF}_{\mathcal{B}}(f) = \{ (0, \xi) ; \xi \in \Gamma \}$$

$$\text{WF}_{\mathcal{B}}(\text{Op}(a)f) = \{ (x_0, \xi) ; \xi \in \Gamma \},$$

which are not overlapping when $x_0 \neq 0$.

Next we apply Theorem 3.1 on operators which are elliptic with respect to $S_{\rho, \delta}^{(\omega_0)}(\mathbf{R}^{2d})$, where $\omega_0 \in \mathcal{P}(\mathbf{R}^{2d})$. More precisely, assume that $0 \leq \delta < \rho \leq 1$ and $a \in S_{\rho, \delta}^{(\omega_0)}(\mathbf{R}^{2d})$. Then a and $\text{Op}(a)$ are called (locally) *elliptic* with respect to $S_{\rho, \delta}^{(\omega_0)}(\mathbf{R}^{2d})$ or ω_0 , if for each compact set $K \subseteq \mathbf{R}^d$, there are positive constants c and R such that

$$|a(x, \xi)| \geq c\omega_0(x, \xi), \quad x \in K, \quad |\xi| \geq R$$

Since $|a(x, \xi)| \leq C\omega_0(x, \xi)$, it follows from the definitions that for each multi-index α , there are constants C_α such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} |a(x, \xi)| \langle \xi \rangle^{-\rho|\beta| + \delta|\alpha|}, \quad x \in K, \quad |\xi| > R.$$

(See e. g. [2, 16].) It immediately follows from the definitions that $\text{Char}_{(\omega_0)}(a) = \emptyset$ when a is elliptic with respect to ω_0 . The following result is now an immediate consequence of Theorem 3.1.

Theorem 3.7. *Let $\omega \in \mathcal{P}(\mathbf{R}^{2d})$, $\omega_0 \in \mathcal{P}_{\rho, 0}(\mathbf{R}^{2d})$, $q \in [1, \infty]$, $\rho > 0$, and let $a \in S_{\rho, 0}^{(\omega_0)}(\mathbf{R}^{2d})$ be elliptic with respect to ω_0 . Also let \mathcal{B} and \mathcal{C} be as in Theorem 3.1. If $f \in \mathcal{S}'(\mathbf{R}^d)$, then*

$$\text{WF}_{\mathcal{C}}(\text{Op}(a)f) = \text{WF}_{\mathcal{B}}(f).$$

Corollary 3.8. *Assume that the hypothesis in Theorem 3.7 is fulfilled with $\omega = \omega_0$, and let E be a parametrix for $\text{Op}(a)$. Then $\varphi E \in \mathcal{F}L_{(\omega)}^\infty$ for every $\varphi \in C_0^\infty(\mathbf{R}^d)$, i. e. for each $x_0 \in$ and $\varphi \in C_0^\infty(\mathbf{R}^d)$, there is a constant C such that (0.1) holds.*

Proof. >From the assumptions it follows that $\text{Op}(a)E = \delta_0 + \varphi$ for some $\varphi \in C^\infty(\mathbf{R}^d)$. The result is then a consequence of

$$\text{WF}_{\mathcal{F}L_{(\omega_0)}^\infty}(E) = \text{WF}_{\mathcal{F}L^\infty}(\text{Op}(a)E) = \text{WF}_{\mathcal{F}L^\infty}(\delta_0 + \varphi) = \emptyset,$$

by Theorem 3.7, where the last equality follows from the fact that $\delta_0 \in \mathcal{F}L^\infty$. \square

Example 3.9. Let $a(x, \xi) = a(\xi)$ be the symbol to the hypoelliptic partial differential operator $\text{Op}(a)$ with constant coefficients (cf. [16, Chapter XI] for strict definitions). Then a is elliptic with respect to

$$\omega(x, \xi) = \omega(\xi) = (1 + |a(\xi)|),$$

which belongs to $\mathcal{P}_{\rho,0}(\mathbf{R}^d)$ for some $\rho > 0$. Hence it follows from Theorem 3.7 and Corollary 3.8 that if $\varphi \in C_0^\infty(\mathbf{R}^d)$, then

$$|(1 + |a(\xi)|)\mathcal{F}(\varphi \cdot E)(\xi)| \in L^\infty(\mathbf{R}^d).$$

An important hypoelliptic operator concerns the heat operator $\partial_t - \Delta_x$, $(x, t) \in \mathbf{R}^{d+1}$, with symbol $a(x, t, \xi, \tau) = |\xi|^2 + i\tau$. In this case, a is elliptic with respect to

$$\omega(x, t, \xi, \tau) = (1 + |\xi|^2 + |\tau|).$$

Hence it follows that

$$(1 + |\xi|^4 + |\tau|^2)^{1/2} |\mathcal{F}(\varphi \cdot E)(\xi, \tau)| \in L^\infty(\mathbf{R}^{d+1}),$$

when $\varphi \in C_0^\infty(\mathbf{R}^{d+1})$.

For the heat operator we note that

$$\text{Char}'(a) = \{ (x, t, 0, \tau) ; x \in \mathbf{R}^d, t \in \mathbf{R}, \tau \neq 0 \},$$

which is not empty (see Remark 1.4 for the definition of $\text{Char}'(a)$). Hence, $\text{Char}_{(\omega_a)}(a)$ is strictly smaller than $\text{Char}'(a)$ in this case.

4. WAVE-FRONT SETS OF SUP AND INF TYPES AND PSEUDO-DIFFERENTIAL OPERATORS

In this section we put the micro-local analysis in a more general context comparing to previous sections, and define wave-front sets with respect to sequences of Fourier Lebesgue type spaces. We also explain some consequences of the investigations in previous sections in this general setting. For example we show how one can obtain micro-local results which involve only classical wave-front sets (cf. Remark 4.2 and Theorem 4.5 below).

Let $\omega_j \in \mathcal{P}(\mathbf{R}^{2d})$ and $q_j \in [1, \infty]$ when j belongs to some index set J , and let \mathcal{B} be the array of spaces, given by

$$(\mathcal{B}_j) \equiv (\mathcal{B}_j)_{j \in J}, \quad \text{where} \quad \mathcal{B}_j = \mathcal{F}L_{(\omega_j)}^{q_j} = \mathcal{F}L_{(\omega_j)}^{q_j}(\mathbf{R}^d), \quad j \in J. \quad (4.1)$$

If $f \in \mathcal{S}'(\mathbf{R}^d)$, and (\mathcal{B}_j) is given by (4.1), then we let $\Theta_{(\mathcal{B}_j)}^{\text{sup}}(f)$ be the set of all $\xi \in \mathbf{R}^d \setminus 0$ such that for some $\Gamma = \Gamma_\xi$ and each $j \in J$ it holds $|f|_{\mathcal{B}_j(\Gamma)} < \infty$. We also let $\Theta_{(\mathcal{B}_j)}^{\text{inf}}(f)$ be the set of all $\xi \in \mathbf{R}^d \setminus 0$ such that for some $\Gamma = \Gamma_\xi$ and some $j \in J$ it holds $|f|_{\mathcal{B}_j(\Gamma)} < \infty$. Finally we let $\Sigma_{(\mathcal{B}_j)}^{\text{sup}}(f)$ and $\Sigma_{(\mathcal{B}_j)}^{\text{inf}}(f)$ be the complements in $\mathbf{R}^d \setminus 0$ of $\Theta_{(\mathcal{B}_j)}^{\text{sup}}(f)$ and $\Theta_{(\mathcal{B}_j)}^{\text{inf}}(f)$ respectively.

Definition 4.1. Let J be an index set, $q_j \in [1, \infty]$, $\omega_j \in \mathcal{P}(\mathbf{R}^{2d})$ when $j \in J$, (\mathcal{B}_j) be as in (4.1), and let X be an open subset of \mathbf{R}^d .

- (1) The wave-front set of $f \in \mathcal{D}'(X)$, of *sup-type* with respect to (\mathcal{B}_j) , $\text{WF}_{(\mathcal{B}_j)}^{\text{sup}}(f)$, consists of all pairs (x_0, ξ_0) in $X \times (\mathbf{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_{(\mathcal{B}_j)}^{\text{sup}}(\varphi f)$ holds for each $\varphi \in C_0^\infty(X)$ such that $\varphi(x_0) \neq 0$;
- (2) The wave-front set of $f \in \mathcal{D}'(X)$, of *inf-type* with respect to (\mathcal{B}_j) , $\text{WF}_{(\mathcal{B}_j)}^{\text{inf}}(f)$ consists of all pairs (x_0, ξ_0) in $X \times (\mathbf{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_{(\mathcal{B}_j)}^{\text{sup}}(\varphi f)$ holds for each $\varphi \in C_0^\infty(X)$ such that $\varphi(x_0) \neq 0$.

Remark 4.2. Let $\omega_j(x, \xi) = \langle \xi \rangle^{-j}$ for $j \in J = \mathbf{N}_0$. Then it follows that $\text{WF}_{(\mathcal{B}_j)}^{\text{sup}}(f)$ in Definition 4.1 is equal to the standard wave front set $\text{WF}(f)$ in Chapter VIII in [16].

The following result follows immediately from Theorems 3.1 and its proof. We omit the details. Here we let

$$\mathcal{C}_j = \mathcal{F}L_{(\omega_j/\omega_0)}^{q_j}(\mathbf{R}^d) \quad \text{and} \quad (\mathcal{C}_j) = (\mathcal{C}_j)_{j \in J}. \quad (4.2)$$

Theorem 3.1'. Let $\rho > 0$, $\omega_j \in \mathcal{P}(\mathbf{R}^{2d})$, $\omega_0 \in \mathcal{P}_{\rho,0}(\mathbf{R}^{2d})$, $a \in S_{\rho,0}^{(\omega_0)}(\mathbf{R}^{2d})$, $f \in \mathcal{S}'(\mathbf{R}^d)$ and $q_j \in [1, \infty]$ for $j \in J$. Also let

$$(\mathcal{B}_j) \equiv (\mathcal{B}_j)_{j \in J} \quad \text{and} \quad (\mathcal{C}_j) \equiv (\mathcal{C}_j)_{j \in J},$$

where

$$\mathcal{B}_j = \mathcal{F}L_{(\omega_j)}^{q_j} = \mathcal{F}L_{(\omega_j)}^{q_j}(\mathbf{R}^d) \quad \text{and} \quad \mathcal{C}_j = \mathcal{F}L_{(\omega_j/\omega_0)}^{q_j} = \mathcal{F}L_{(\omega_j/\omega_0)}^{q_j}(\mathbf{R}^d).$$

Then

$$\begin{aligned} \text{WF}_{(\mathcal{C}_j)}^{\text{sup}}(\text{Op}(a)f) &\subseteq \text{WF}_{(\mathcal{B}_j)}^{\text{sup}}(f) \\ &\subseteq \text{WF}_{(\mathcal{C}_j)}^{\text{sup}}(\text{Op}(a)f) \cup \text{Char}_{(\omega_0)}(a), \end{aligned} \quad (3.1)'$$

and

$$\begin{aligned} \text{WF}_{(\mathcal{C}_j)}^{\text{inf}}(\text{Op}(a)f) &\subseteq \text{WF}_{(\mathcal{B}_j)}^{\text{inf}}(f) \\ &\subseteq \text{WF}_{(\mathcal{C}_j)}^{\text{inf}}(\text{Op}(a)f) \cup \text{Char}_{(\omega_0)}(a). \end{aligned} \quad (3.1)''$$

Remark 4.3. We note that many properties valid for the wave-front sets of Fourier Lebesgue type also hold for wave-front sets in the present section. For example, the conclusions in Remark 3.6 and Theorem 3.7 hold for wave-front sets of sup- and inf-types.

There are (somewhat technical) generalizations of Theorems 3.1 and 3.1' to pseudo-differential operators with symbols in $S_{\rho,\delta}^{(\omega_0)}$, when $0 \leq \delta < \rho \leq 1$. For example, when generalizing Theorem 3.1' to $\delta \geq 0$ the key estimate (3.6) needs to be modified into

$$|(\mathcal{F}_1 a)(\xi, \eta)| \leq C_N \langle \xi \rangle^{-N} \langle \eta \rangle^{\delta N} \omega_0(\eta).$$

This in turn implies that in (3.1)' and (3.1)'', the array (\mathcal{C}_j) on the left-hand (right-hand) side embeddings should be replaced by $(\mathcal{F}L_{(\omega_j, -N_j/\omega_0)}^{q_j})$, and the right-hand side embeddings by $(\mathcal{F}L_{(\omega_j, N_j/\omega_0)}^{q_j})$. Here

$$\omega_{j,s}(x, \xi) = \omega_j(x, \xi) \langle \xi \rangle^s,$$

and

$$N_j = \delta(t_j + C), \quad (4.3)$$

where C should be chosen large enough and depends on the order of the involved distribution f in (3.1)' and (3.1)'' and the dimension d , and t_j is chosen such that the inequality

$$\omega_j(x, \xi_1 + \xi_2) \leq C \omega_j(x, \xi_1) \langle \xi_2 \rangle^{t_j}, \quad (4.4)$$

should hold.

The following generalization of Theorem 3.1' is obtained by modifying the proof of Proposition 3.3 and Theorem 3.1'. The details are left for the reader.

Theorem 4.4. *Let $0 \leq \delta < \rho \leq 1$, $\omega_j \in \mathcal{P}(\mathbf{R}^{2d})$, $\omega_0 \in \mathcal{P}_{\rho, \delta}(\mathbf{R}^{2d})$, $a \in S_{\rho, 0}^{(\omega_0)}(\mathbf{R}^{2d})$, $f \in \mathcal{S}'(\mathbf{R}^d)$ and $q_j \in [1, \infty]$ for $j \in J$. Also let N_j be given by (4.3) with C only depending on the order of f and the dimension d ,*

$$(\mathcal{B}_j) \equiv (\mathcal{B}_j)_{j \in J} \quad \text{and} \quad (\mathcal{C}_j^\pm) \equiv (\mathcal{C}_j^\pm)_{j \in J},$$

where

$$\mathcal{B}_j = \mathcal{F}L_{(\omega_j)}^{q_j} = \mathcal{F}L_{(\omega_j)}^{q_j}(\mathbf{R}^d) \quad \text{and} \quad \mathcal{C}_j^\pm = \mathcal{F}L_{(\omega_j, \pm N_j/\omega_0)}^{q_j} = \mathcal{F}L_{(\omega_j, \pm N_j/\omega_0)}^{q_j}(\mathbf{R}^d).$$

Then

$$\begin{aligned} \text{WF}_{(\mathcal{C}_j^-)}^{\sup}(\text{Op}(a)f) &\subseteq \text{WF}_{(\mathcal{B}_j)}^{\sup}(f) \\ &\subseteq \text{WF}_{(\mathcal{C}_j^+)}^{\sup}(\text{Op}(a)f) \cup \text{Char}_{(\omega_0)}(a), \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} \text{WF}_{(\mathcal{C}_j^-)}^{\inf}(\text{Op}(a)f) &\subseteq \text{WF}_{(\mathcal{B}_j)}^{\inf}(f) \\ &\subseteq \text{WF}_{(\mathcal{C}_j^+)}^{\inf}(\text{Op}(a)f) \cup \text{Char}_{(\omega_0)}(a), \end{aligned} \quad (4.5)'$$

provided C in (4.3) is chosen large enough.

A combination of Remark 4.3 and Theorem 4.4 now gives the following result concerning wave-front sets of Hörmander type.

Theorem 4.5. *Let $0 \leq \delta < \rho \leq 1$ and $\omega_0 \in \mathcal{P}_{\rho, \delta}(\mathbf{R}^{2d})$. For every $f \in \mathcal{S}'(\mathbf{R}^d)$ and $a \in S_{\rho, \delta}^{(\omega_0)}(\mathbf{R}^{2d})$ it holds*

$$\text{WF}(\text{Op}(a)f) \subseteq \text{WF}(f) \subseteq \text{WF}(\text{Op}(a)f) \cup \text{Char}_{(\omega_0)}(a).$$

In particular, if in addition a is elliptic with respect to ω_0 , then

$$\text{WF}(\text{Op}(a)f) = \text{WF}(f).$$

5. WAVE FRONT SETS WITH RESPECT TO MODULATION SPACES

In this section we define wave-front sets with respect to modulation spaces, and show that they coincide with wave-front sets of Fourier Lebesgue types. In particular, any property valid for wave-front set of Fourier Lebesgue type carry over to wave-front set of modulation space type.

Let $\phi \in \mathcal{S}(\mathbf{R}^d) \setminus 0$, $\omega \in \mathcal{P}(\mathbf{R}^{2d})$, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone and let $p, q \in [1, \infty]$. For any $f \in \mathcal{S}'(\mathbf{R}^d)$ we set

$$|f|_{\mathcal{B}(\Gamma)} = |f|_{\mathcal{B}(\phi, \Gamma)} \equiv \left(\int_{\Gamma} \left(\int_{\mathbf{R}^d} |V_{\phi} f(x, \xi) \omega(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q}$$

when $\mathcal{B} = M_{(\omega)}^{p,q} = M_{(\omega)}^{p,q}(\mathbf{R}^d)$ (5.1)

(with obvious interpretation when $p = \infty$ or $q = \infty$). We note that $|\cdot|_{\mathcal{B}(\Gamma)}$ defines a semi-norm on \mathcal{S}' which might attain the value $+\infty$. If $\Gamma = \mathbf{R}^d \setminus 0$, then $|f|_{\mathcal{B}(\Gamma)} = \|f\|_{M_{(\omega)}^{p,q}}$. We also set

$$|f|_{\mathcal{B}(\Gamma)} = |f|_{\mathcal{B}(\phi, \Gamma)} \equiv \left(\int_{\mathbf{R}^d} \left(\int_{\Gamma} |V_{\phi} f(x, \xi) \omega(x, \xi)|^q d\xi \right)^{p/q} dx \right)^{1/p}$$

when $\mathcal{B} = W_{(\omega)}^{p,q} = W_{(\omega)}^{p,q}(\mathbf{R}^d)$ (5.2)

and note that similar properties hold for this semi-norm comparing to (5.1).

Let $\omega \in \mathcal{P}(\mathbf{R}^{2d})$, $p, q \in [1, \infty]$, $f \in \mathcal{D}'(X)$, and let $\mathcal{B} = M_{(\omega)}^{p,q}$ or $\mathcal{B} = W_{(\omega)}^{p,q}$. Then $\Theta_{\mathcal{B}}(f)$, $\Sigma_{\mathcal{B}}(f)$ and the wave-front set $\text{WF}_{\mathcal{B}}(f)$ of f with respect to the modulation space \mathcal{B} are defined in the same way as in Section 2, after replacing the semi-norms of Fourier Lebesgue types in (2.2) with the semi-norms in (5.1) or (5.2).

The following result shows that wave-front sets of Fourier Lebesgue and modulation space types agree with each others.

Theorem 5.1. *Let $p, q \in [1, \infty]$, an open set $X \subseteq \mathbf{R}^d$, $\omega \in \mathcal{P}(\mathbf{R}^{2d})$, $\mathcal{B} = \mathcal{FL}_{(\omega)}^q(\mathbf{R}^d)$, and let $\mathcal{C} = M_{(\omega)}^{p,q}(\mathbf{R}^d)$ or $W_{(\omega)}^{p,q}(\mathbf{R}^d)$. If $f \in \mathcal{D}'(X)$, then*

$$\text{WF}_{\mathcal{B}}(f) = \text{WF}_{\mathcal{C}}(f). \quad (5.3)$$

In particular, $\text{WF}_{\mathcal{C}}(f)$ is independent of p and $\phi \in \mathcal{S}(\mathbf{R}^d) \setminus 0$ in (5.1) and (5.2).

Proof. We only consider the case $\mathcal{C} = M_{(\omega)}^{p,q}$. The case $\mathcal{C} = W_{(\omega)}^{p,q}$ follows by similar arguments and is left for the reader. We may also assume that $f \in \mathcal{E}'(\mathbf{R}^d)$ and that $\omega(x, \xi) = \omega(\xi)$, since the statements only involve local assertions.

First we prove that $\text{WF}_C(f)$ is independent of $\phi \in \mathcal{S}(\mathbf{R}^d) \setminus 0$. Therefore assume that $\phi, \phi_1 \in \mathcal{S} \setminus 0$ and let $|\cdot|_{C_1(\Gamma)}$ be the semi-norm in (5.1) after ϕ has been replaced by ϕ_1 . Let Γ_1 and Γ_2 be open cones in \mathbf{R}^d such that $\overline{\Gamma_2} \subseteq \Gamma_1$. The asserted independency of ϕ follows if we prove that

$$|f|_{C(\Gamma_2)} \leq C(|f|_{C_1(\Gamma_1)} + 1), \quad (5.4)$$

for some constant C .

When proving (5.4) we shall mainly follow the proof of (2.5). Let $v \in \mathcal{P}$ be chosen such that ω is v -moderate, let

$$\Omega_1 = \{(x, \xi); \xi \in \Gamma_1\} \subseteq \mathbf{R}^{2d} \quad \text{and} \quad \Omega_2 = \mathbb{C}\Omega_1 \subseteq \mathbf{R}^{2d},$$

with characteristic functions χ_1 and χ_2 respectively, and set $F_k(x, \xi) = |V_{\phi_1}f(x, \xi)|\omega(\xi)\chi_k(x, \xi)$, $k = 1, 2$, and $G = |V_{\phi}\phi_1(x, \xi)|v(\xi)$. By Lemma 1.7, and the fact that ω is v -moderate we get

$$|V_{\phi}f(x, \xi)\omega(x, \xi)| \leq C((F_1 + F_2) * G)(x, \xi),$$

for some constant C , which implies that

$$|f|_{C(\Gamma_2)} \leq C(J_1 + J_2), \quad (5.5)$$

where

$$J_k = \left(\int_{\Gamma_2} \left(\int |(F_k * G)(x, \xi)|^p dx \right) d\xi \right)^{1/q}, \quad k = 1, 2.$$

Since G turns rapidly to zero at infinity, Young's inequality gives

$$J_1 \leq \|F_1 * G\|_{L_1^{p,q}} \leq \|G\|_{L^1} \|F_1\|_{L_1^{p,q}} = C|f|_{C_1(\Gamma_1)}, \quad (5.6)$$

where $C = \|G\|_{L^1} < \infty$.

Next we consider J_2 . By Lemma 1.6 and the proof of (2.5), it follows that for every $N \geq 0$ there are constants C_N such that

$$F_2(x, \xi) \leq C_N \langle x \rangle^{-N} \langle \xi \rangle^{N_0}, \quad \text{and} \quad \langle \xi - \eta \rangle^{-2N} \leq C_N \langle \xi \rangle^{-N} \langle \eta \rangle^{-N}$$

when $\xi \in \Gamma_2$ and $\eta \in \mathbb{C}\Gamma_1$. This in turn implies that for every $N \geq 0$ there are constants C_N such that

$$(F_2 * G)(x, \xi) \leq C_N \langle x \rangle^{-N} \langle \xi \rangle^{-N}, \quad \xi \in \Gamma_2.$$

Consequently, $J_2 < \infty$. The estimate (5.4) is now a consequence of (5.5), (5.6) and the fact that $J_2 < \infty$. This proves that $\text{WF}_C(f)$ is independent of $\phi \in \mathcal{S}(\mathbf{R}^d) \setminus 0$.

In order to prove (5.3) we assume from now on that ϕ in (5.1) has compact support. We choose $p_0, p_1 \in [1, \infty]$ such that $p_0 \leq p$ and $1/p_1 + 1/p_0 = 1 + 1/p$, and we set $\mathcal{C}_0 = M_{(\omega)}^{p_0, q}$. The result follows if we prove

$$\Theta_{\mathcal{C}_0}(f) \subseteq \Theta_{\mathcal{B}}(f) \subseteq \Theta_C(f) \quad \text{when } p_0 = 1, p = \infty, \quad (5.7)$$

and

$$\Theta_C(f) \subseteq \Theta_{\mathcal{C}_0}(f). \quad (5.8)$$

We start to prove (5.7). We have

$$\begin{aligned}
|f|_{\mathcal{B}(\Gamma)} &\leq C_1 \left(\int_{\Gamma} |\widehat{f}(\xi) \omega(\xi)|^q d\xi \right)^{1/q} \\
&= C_2 \left(\int_{\Gamma} |\mathcal{F} \left(f \int_{\mathbf{R}^d} \phi(\cdot - x) dx \right) (\xi) \omega(\xi)|^q d\xi \right)^{1/q} \\
C_2 &\leq \left(\int_{\Gamma} \left(\int_{\mathbf{R}^d} |\mathcal{F}(f \phi(\cdot - x))(\xi) \omega(\xi)| dx \right)^q d\xi \right)^{1/q} \\
&\leq C_3 \left(\int_{\Gamma} \left(\int_{\mathbf{R}^d} |V_{\phi} f(x, \xi) \omega(\xi)| dx \right)^q d\xi \right)^{1/q} = C_3 |f|_{\mathcal{C}_0(\Gamma)}
\end{aligned}$$

for some constants C_1 , C_2 and C_3 . This gives the first inclusion in (5.7).

Next we prove the second inclusion in (5.7). Let $K \subseteq \mathbf{R}^d$ be compact and chosen such that $V_{\phi} f(x, \xi) = 0$ outside K . This is possible since both f and ϕ have compact supports. Then

$$\begin{aligned}
|f|_{\mathcal{C}(\Gamma_2)} &= \left(\int_{\Gamma_2} \sup_{x \in K} |V_{\phi} f(x, \xi) \omega(x, \xi)|^q d\xi \right)^{1/q} \\
&\leq C_1 \left(\int_{\Gamma_2} \sup_{x \in \mathbf{R}^d} |(|\widehat{f}| * |\mathcal{F}(\phi(\cdot - x))|)(\xi) \omega(\xi)|^q d\xi \right)^{1/q} \\
&= C_1 \left(\int_{\Gamma_2} |(|\widehat{f}| * |\widehat{\phi}|)(\xi) \omega(\xi)|^q d\xi \right)^{1/q} \\
&\leq C_2 \left(\int_{\Gamma_2} (|\widehat{f} \cdot \omega| * |\widehat{\phi} \cdot v|)(\xi))^q d\xi \right)^{1/q},
\end{aligned}$$

for some positive constants C_1 , C_2 and C_3 . By combining the latter estimates with (2.5) and its proof it now follows that for each $N \geq 0$ there are constants C_N such that (2.5) holds with $\varphi = 1$ and $\mathcal{B}_0 = M_{(\omega)}^{p,q}$. Since f has compact support it follows that the right-hand side of (2.5) is finite when $|f|_{\mathcal{B}(\Gamma_1)} < \infty$, provided N is chosen large enough. This proves (5.7).

It remains to prove (5.8). Let K be as above. By Hölder's inequality we get

$$\begin{aligned}
|f|_{\mathcal{C}_0(\Gamma)} &= \left(\int_{\Gamma} \left(\int_{\mathbf{R}^d} |V_{\phi} f(x, \xi) \omega(x, \xi)|^{p_0} dx \right)^{q/p_0} d\xi \right)^{1/q} \\
&\leq C_K \left(\int_{\Gamma} \left(\int_{\mathbf{R}^d} |V_{\phi} f(x, \xi) \omega(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} = C_K |f|_{\mathcal{C}(\Gamma)}.
\end{aligned}$$

This gives (5.8), and the proof is complete. \square

Corollary 5.2. *Let $f \in \mathcal{E}'(\mathbf{R}^d)$, and let \mathcal{B} be equal to $\mathcal{F}L_{(\omega)}^q$, $M_{(\omega)}^{p,q}$ or $W_{(\omega)}^{p,q}$. Then*

$$f \in \mathcal{B} \iff \text{WF}_{\mathcal{B}}(f) = \emptyset.$$

In particular, we recover Theorem 2.1 and Remark 4.4 in [22].

6. WAVE-FRONT SETS AND PSEUDO-DIFFERENTIAL OPERATORS WITH NON-SMOOTH SYMBOLS

In this section we generalize wave-front results in Section 3 to pseudo-differential operators with symbols in $\mathcal{U}_{(\omega)}^{s,\rho}(\mathbf{R}^{2d})$ (see Definition 1.11). In order to state the results we use the convention

$$(\vartheta_1, \vartheta_2) \lesssim (\omega_1, \omega_2) \quad (6.1)$$

when $\omega_j, \vartheta_j \in \mathcal{P}(\mathbf{R}^{2d})$ for $j = 1, 2$ satisfy $\vartheta_j \leq C\omega_j$ for some constant C . We recall that

$$\omega_j(x, \xi_1 + \xi_2) \leq C\omega_j(x, \xi_1)\langle \xi_2 \rangle^{t_j}, \quad j = 1, 2, \quad (6.2)$$

for some positive constants C , t_1 and t_2 , which are independent of $x, \xi_1, \xi_2 \in \mathbf{R}^d$. We let $\omega_{s,\rho}$ be the same as in (1.13) and we use the notation $\omega \preccurlyeq (\omega_1, \omega_2)$ when (1.12) holds for some constant C .

Theorem 6.1. *Let $q \in [1, \infty]$, $\omega_j, \vartheta_j \in \mathcal{P}(\mathbf{R}^{2d})$, $\omega \in \mathcal{P}_{\rho,0}(\mathbf{R}^{4d})$, $0 < \rho \leq 1$, and let $t_j \geq 0$, $j = 1, 2$ be chosen such that (6.1) and (6.2) holds. Also let $\mathcal{B} = \mathcal{F}L_{(\omega_1)}^q$ and $\mathcal{C} = \mathcal{F}L_{(\omega_2)}^q$. Moreover, assume that $\omega_{s,\rho}$ in (1.13) satisfy $\omega_{s,\rho} \preccurlyeq (\omega_1, \omega_2)$, $\omega_{s,\rho} \preccurlyeq (\vartheta_1, \vartheta_2)$ for some choices of*

$$s_1 \geq 0, \quad s_2 \in \mathbf{N}, \quad s_3 > t_1 + t_2 + 2d, \quad \text{and} \quad s_4 \in \mathbf{R}.$$

If $a \in \mathcal{U}_{(\omega)}^{s,\rho}(\mathbf{R}^{2d})$ and $f \in M_{(\vartheta_1)}^\infty(\mathbf{R}^d)$, then

$$\text{WF}_C(\text{Op}(a)f) \subseteq \text{WF}_{\mathcal{B}}(f).$$

By Proposition 1.10, it follows that $\text{Op}(a)f$ in Theorem 6.1 makes sense as an element in $M_{(\vartheta_2)}^\infty(\mathbf{R}^d)$, which contains each space $M_{(\omega_2)}^{p,q}(\mathbf{R}^d)$.

When proving Theorem 6.1, we shall mainly follow the ideas in the proof of Theorem 3.1, and prove some preparing results. The first one of these results can be considered as a generalization of Proposition 3.2 in the case $\delta = 0$.

Proposition 6.2. *Let $a \in \mathcal{U}_{(\omega)}^{s,\rho}(\mathbf{R}^{2d})$, where $\omega \in \mathcal{P}_{\rho,0}(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$, $0 \leq s_1$, $0 \leq s_2 \in \mathbf{Z}$, and let L_a be the operator from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$ which is given by (3.2). Then the following is true:*

- (1) $L_a = \text{Op}(a_0)$, for some $a_0 \in \mathcal{S}'(\mathbf{R}^{2d})$ such that

$$\partial_\xi^\alpha a_0(x, \xi) \in \bigcap_{s_4 \geq 0} M_{(1/\omega_{s,\rho})}^{\infty,1}(\mathbf{R}^{2d}), \quad (6.3)$$

for all multi-indices α such that $|\alpha| \leq 2s_2$;

- (2) *if $p, q \in [1, \infty]$, and $\omega_1, \omega_2 \in \mathcal{P}(\mathbf{R}^{2d})$ fulfill $\omega_{s,\rho} \preccurlyeq (\omega_1, \omega_2)$, then the definition of L_a extends uniquely to a continuous map from $M_{(\omega_1)}^{p,q}(\mathbf{R}^d)$ to $M_{(\omega_2)}^{p,q}(\mathbf{R}^d)$.*

Proof. We use the same notations as in the proof of Proposition 3.2, with the difference that $s = (s_1, s_2, s_3, s_4) \in \mathbf{R}^4$.

(1) From the proof of Proposition 3.2 it follows that $L_a = \text{Op}(b_s)$, where b_s is given by (3.3). By the support properties of φ_1 , φ_2 and φ , it follows from Proposition 4.3 in [22] that

$$b_s \in M_{(\nu_{s,t,\rho})}^{\infty,1}(\mathbf{R}^{3d}), \quad \text{for every } s_4, t \geq 0,$$

where

$$\begin{aligned} \nu_{s,t,\rho}(x, y, \xi, \zeta, \eta, z) &= \omega_{s,\rho}(x, \xi, \zeta, z)^{-1} \kappa(x, y, \xi, \zeta, \eta, z) \\ &= \omega(x, \xi, \zeta, z)^{-1} \langle x \rangle^{s_4} \langle \zeta \rangle^{s_3} \langle x - y \rangle^{2s_2} \langle \xi \rangle^{\rho s_2} \langle \eta \rangle^t \langle z \rangle^{s_1}. \end{aligned}$$

Here

$$\kappa(x, y, \zeta, \xi, \eta, z) = \langle z - y \rangle^{2s_2} \langle \eta \rangle^t$$

In view of Sections 18.1 and 18.2 in [16] it follows that $\text{Op}(b_s) = \text{Op}(a_0)$ when

$$c_s(x, y, \xi) = e^{i\langle D_\xi, D_y \rangle} b_s(x, y, \xi) \quad \text{and} \quad a_0(x, \xi) = c_s(x, x, \xi).$$

We have to prove that a_0 is well-defined and fulfills (6.3) when $|\alpha| \leq 2s_2$. By Proposition 1.7 in [29] we have

$$c_s \in M_{(\tilde{\nu}_{s,t,\rho})}^{\infty,1}(\mathbf{R}^{3d}), \quad \text{for every } s_4, t \geq 0, \quad (6.4)$$

where

$$\begin{aligned} \tilde{\nu}_{s,t,\rho}(x, y, \xi, \zeta, \eta, z) &= \nu_{s,t,\rho}(x, y - z, \xi - \eta, \zeta, \eta, z) \\ &= \omega(x, \xi - \eta, \zeta, z)^{-1} \langle x \rangle^{s_4} \langle \zeta \rangle^{s_3} \langle y - z - x \rangle^{2s_2} \langle \xi - \eta \rangle^{2\rho s_2} \langle \eta \rangle^t \langle z \rangle^{s_1} \end{aligned}$$

Since $\omega \in \mathcal{P}$, the right-hand side can be estimated by

$$C \omega_{s,\rho}(x, \xi, \zeta, z)^{-1} \langle y - z \rangle^{2s_2} \langle \eta \rangle^t = C \omega_{s,\rho}(x, \xi, \zeta, z)^{-1} \kappa(x, y, \xi, \zeta, \eta, z),$$

for some constant C , provided s_4 and t have been replaced by larger constants if necessary. Since (6.4) holds for any $s_4 \geq 0$ and $t \geq 0$ we get

$$c_s \in M_{(1/\omega_{s,t,\rho})}^{\infty,1}(\mathbf{R}^{3d}), \quad \text{for every } s_4, t \geq 0,$$

where

$$\omega_{s,t,\rho}(x, y, \xi, \zeta, \eta, z) = \omega_{s,\rho}(x, \xi, \zeta, z) / \kappa(x, y, \xi, \zeta, \eta, z).$$

From the fact that

$$\sup_{z,\eta} \left(\left(\inf_{x,\zeta} \kappa(x, y, \xi, \zeta, \eta, z) \right)^{-1} \right) = 1 < \infty,$$

it follows now by Theorem 3.2 in [28] that $a_0(x, \xi) = c_s(x, x, \xi)$ is well-defined and belongs to $M_{(1/\omega_{s,\rho})}^{\infty,1}(\mathbf{R}^{2d})$, for each $s_4 \geq 0$. This proves (6.3) in the case $\alpha = 0$.

If we let $b_{s,\alpha}$ for $|\alpha| \leq 2s_2$ here above be defined by

$$b_{s,\alpha}(x, y, \xi) = \partial_\xi^\alpha a_0(x, \xi) \varphi_1(x) \varphi_2(y),$$

then

$$\text{Op}(\partial^\alpha a_0) = \text{Op}(b_{s,\alpha}), \quad |\alpha| \leq 2s_2.$$

By similar arguments as in the first part of the proof it follows that $\partial^\alpha a_0 \in M_{(1/\omega_s, \rho)}^{\infty, 1}(\mathbf{R}^{2d})$, for each $s_4 \geq 0$. The details are left for the reader. This proves (1), and the assertion (2) is an immediate consequence of (1) and Proposition 1.10. The proof is complete. \square

Next we consider properties of the wave-front set of $\text{Op}(a)f$ at a fixed point when f is concentrated to that point. In these considerations it is natural to assume that involved weight functions satisfy

$$\begin{aligned} \omega_j(x, \xi) &= \omega_j(\xi), & \vartheta_j(x, \xi) &= \vartheta_j(\xi), \quad j = 1, 2, \\ \omega(0, \xi, \zeta, z) &\leq C \frac{\omega_1(\xi + \zeta)}{\omega_2(\xi)}, & \omega(0, \xi, \zeta, z) &\leq C \frac{\vartheta_1(\xi + \zeta)}{\vartheta_2(\xi)}, \end{aligned} \quad (6.5)$$

and we set

$$\omega_s(x, \xi, \zeta, z) = \langle x \rangle^{-s_4} \langle \zeta \rangle^{-s_3} \omega(0, \xi, \zeta, z), \quad s \in \mathbf{R}^4. \quad (6.6)$$

We also note that

$$\omega_j(\xi_1 + \xi_2) \leq C \omega_j(\xi_1) \langle \xi_2 \rangle^{t_j}, \quad j = 1, 2, \quad (6.7)$$

for some real numbers t_1 and t_2 .

Proposition 6.3. *Let $q \in [1, \infty]$, and let $\omega_s \in \mathcal{P}(\mathbf{R}^{4d})$, $\omega \in \mathcal{P}(\mathbf{R}^{3d})$, $\omega_j, \vartheta_j \in \mathcal{P}(\mathbf{R}^d)$, $j = 1, 2$, fulfill $(\vartheta_1, \vartheta_2) \lesssim (\omega_1, \omega_2)$, (6.5)–(6.7), $s_4 > d$ and*

$$s_3 > t_1 + t_2 + 2d.$$

If $a \in M_{(1/\omega_s)}^{\infty, 1}(\mathbf{R}^{2d})$ and $f \in M_{(\vartheta_1)}^\infty(\mathbf{R}^d) \cap \mathcal{E}'(\mathbf{R}^d)$ then (1) and (2) in Proposition 3.3 holds for $\mathcal{B} = \mathcal{F}L_{(\omega_1)}^q$ and $\mathcal{C} = \mathcal{F}L_{(\omega_2)}^q$.

Proof. As for the proof of Proposition 3.3, we only prove the result for $q < \infty$. The slight modifications to the case $q = \infty$ are left for the reader. We also use similar notations as in the proof of Proposition 3.3.

Let $\phi_1, \phi_2 \in C_0^\infty(\mathbf{R}^d)$ be such that

$$\int_{\mathbf{R}^d} \phi_1(x) dx = 1, \quad \phi_2(0) = (2\pi)^{-d/2},$$

and set $\phi = \phi_1 \otimes \phi_2$. It follows from Fourier's inversion formula that

$$\mathcal{F}_1 a(\xi, \eta) = \iint (V_\phi a)(x, \eta, \xi, z) e^{i\langle z, \eta \rangle} dx dz.$$

By Remark 1.9 it follows that

$$\begin{aligned}
|(\mathcal{F}_1 a)(\xi - \eta, \eta) \omega_2(\xi)| &\leq C_1 \iint |(V_\phi a)(x, \eta, \xi - \eta, z) \omega_2(\xi)| dx dz \\
&\leq C_2 \iint \left(\sup_{\eta} \left(\sup_{x, \xi \in \mathbf{R}^d} |V_\phi a(x, \xi, \zeta, z) \omega(x, \xi, \zeta, z)^{-1}| \right) \right) \times \\
&\quad \times \langle x \rangle^{-s_4} \langle \xi - \eta \rangle^{-s_3} \omega(0, \eta, \xi - \eta, z) \omega_2(\xi) dx dz \\
&\leq C_3 \|a\|_{\widetilde{M}_{(1/\omega_s)}} \langle \xi - \eta \rangle^{-s_3} \left(\sup_z \omega(0, \eta, \xi - \eta, z) \right) \omega_2(\xi) \\
&\leq C_4 \|a\|_{M_{(1/\omega_s)}^{\infty, 1}} \langle \xi - \eta \rangle^{-s_3} \left(\sup_z \omega(0, \eta, \xi - \eta, z) \right) \omega_2(\xi) \\
&\leq C_5 \|a\|_{M_{(1/\omega_s)}^{\infty, 1}} \langle \xi - \eta \rangle^{-s_3} \omega_1(\eta),
\end{aligned}$$

for some constants C_1, \dots, C_5 . The result now follows by similar arguments as in the proof of Proposition 3.3, after replacing (3.7) with the latter estimates. The details are left for the reader, and the proof is complete. \square

Proof of Theorem 6.1. Let $\varphi \in C_0^\infty(\mathbf{R}^d)$ be such that $\varphi = 1$ in a neighborhood of x_0 , and let $\varphi_2 = 1 - \varphi$. If $(x_0, \xi_0) \notin \text{WF}_B(f)$, then it follows from Proposition 6.2 that

$$(x_0, \xi_0) \notin \text{WF}_C(\text{Op}(a)(\varphi_2 f)).$$

Since

$$\text{WF}_C(\text{Op}(a)f) \subseteq \text{WF}_C(\text{Op}(a)(\varphi f)) \cup \text{WF}_C(\text{Op}(a)(\varphi_2 f)),$$

the result follows if we prove that

$$(x_0, \xi_0) \notin \text{WF}_C(\text{Op}(a_0)(\varphi f)), \quad (6.8)$$

where $a_0(x, \xi) = \varphi(x)a(x, \xi)$.

We may assume that $\omega(x, \xi, \zeta, z)$, $\omega_j(x, \xi)$ and $\vartheta_j(x, \xi)$ are independent of the x variable when proving (6.8), since both φf and a_0 are compactly supported with respect to the x variable. The result is now an immediate consequence of Proposition 6.3. \square

Remark 6.4. Let $\mathcal{U}_{(\omega)}^{s, \rho, t}(\mathbf{R}^{2d})$ be as $\mathcal{U}_{(\omega)}^{s, \rho}(\mathbf{R}^{2d})$, after $\omega_{s, \rho}(x, \xi, \zeta, z)$ has been replaced by

$$\omega_{s, t, \rho}(x, \xi, \zeta, z) = \omega_{s, \rho}(x + tz, \xi + t\zeta, \zeta, z), \quad t \in \mathbf{R},$$

in the definition of $\mathcal{U}_{(\omega)}^{s, \rho}(\mathbf{R}^{2d})$. Then it follows from Proposition 1.7 in [29] that if $a \in \mathcal{U}_{(\omega)}^{s, \rho}(\mathbf{R}^{2d})$, then Theorem 6.1 remains valid after $\omega(x, \xi, \zeta, z)$ has been replaced by $\omega(x + tz, \xi + t\zeta, \zeta, z)$ and $\text{Op}(a)$ has been replaced by $\text{Op}_t(a)$.

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